Computational Social Choice

From Arrow's impossibility to Fishburn's maximal lotteries

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St. Petersburg, Russia, October 2016

DSS
Decision Sciences & Systems

Motivation

- *What is "social choice theory"?*
  - How to aggregate possibly conflicting preferences into collective choices in a fair and satisfactory way?
  - Origins: mathematics, economics, and political science
  - Essential ingredients
    - Autonomous agents (e.g., human or software agents)
    - A set of alternatives (depending on the application, alternatives can be political candidates, resource allocations, coalition structures, etc.)
    - Preferences over alternatives
    - Aggregation functions
- The axiomatic method will play a crucial role in this tutorial.
  - Which formal properties should an aggregation function satisfy?
  - Which of these properties can be satisfied simultaneously?

Handbook of Computational Social Choice
(Cambridge University Press, 2016)

1. Introduction to Computational Social Choice

Part I: Voting

2. Introduction to the Theory of Voting
3. Tournament Solutions
4. Weighted Tournament Solutions
5. Dodgson's Rule and Young's Rule
6. Barriers to Manipulation in Voting
7. Control and Bribery in Voting
8. Rationalizations of Voting Rules
9. Voting in Combinatorial Domains
10. Incomplete Information and Communication in Voting

Part II: Fair Allocation

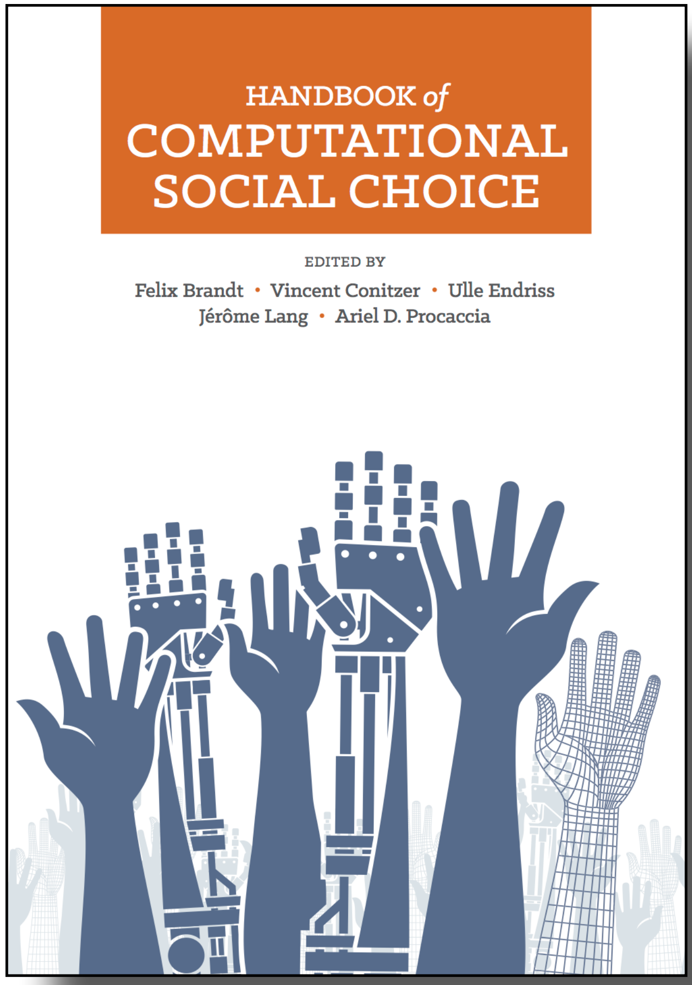
11. Introduction to the Theory of Fair Allocation
12. Fair Allocation of Indivisible Goods
13. Cake Cutting Algorithms

Part III: Coalition Formation

14. Matching under Preferences
15. Hedonic Games
16. Weighted Voting Games

Part IV: Additional Topics

17. Judgment Aggregation
18. The Axiomatic Approach and the Internet
19. Knockout Tournaments



Syllabus

- Introductory examples
- Rational choice theory
- Arrow's impossibility
- Tournament solutions
- Computer-aided theorem proving
- Probabilistic social choice

Plurality

- Why are there different voting rules?
  - What's wrong with plurality (the most widespread voting rule) where alternatives that are ranked first by most voters win?
  - Consider a *preference profile* with 21 voters, who rank four alternatives as in the table on the right.

3	**5**	**7**	**6**
a	*a*	*b*	*c*
b	*c*	*d*	*b*
c	*b*	*c*	*d*
d	*d*	*a*	*a*

Example due to Condorcet (1785)

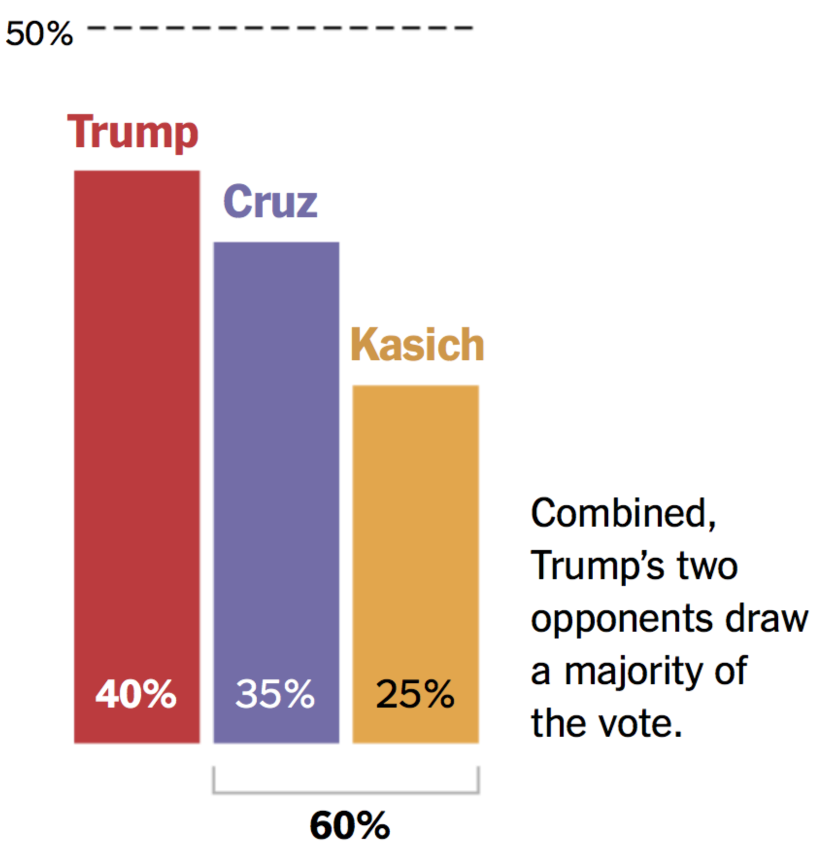
  - Alternative *a* is the unique plurality winner despite
    - a majority of voters think *a* is the worst alternative,
    - *a* loses against *b*, *c*, and *d* in pairwise majority comparisons, and
    - if the preferences of all voters are reversed, *a* still wins.
  - In July 2010, 22 experts on social choice theory met in France and voted on which voting rules should be used. Plurality received *no support* at all (among 18 rules).

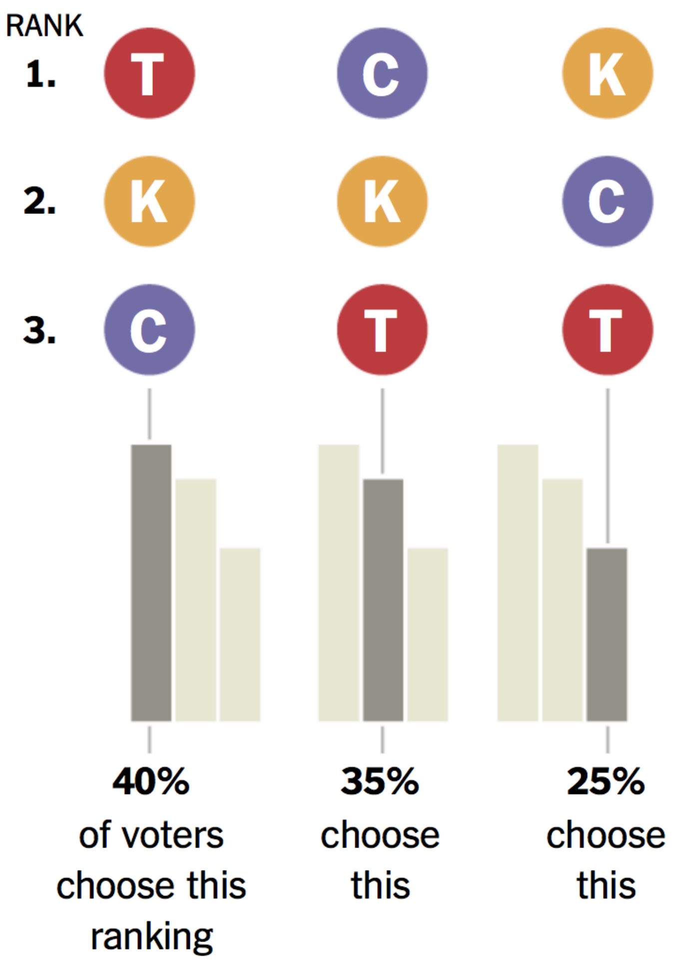
Eric S. Maskin

More Timely Example

Amartya K. Sen

- *How majority rule might have stopped Donald Trump* (Eric Maskin and Amartya Sen, New York Times, April 2016)





5 Common Voting Rules

- Plurality
  - Used in most democratic countries, ubiquitous
  - Alternatives that are ranked first by most voters
- Borda
  - Used in Slovenia, academic institutions, Eurovision song contest
  - The most preferred alternative of each voter gets m-1 points, the second most-preferred m-2 points, etc. Alternatives with highest accumulated score win.
- Plurality with runoff
  - Used to elect the President of France
  - The two alternatives that are ranked first by most voters face off in a majority runoff.

5 Common Voting Rules (ctd.)

- Instant-runoff
  - Used in Australia, Ireland, Malta, Academy awards
  - Alternatives that are ranked first by the lowest number of voters are deleted. Repeat until no more alternatives can be deleted. The remaining alternatives win.
  - In the UK 2011 alternative vote referendum, people chose plurality over instant-runoff.
- Sequential majority comparisons
  - Used by US congress to pass laws (aka *amendment procedure*) and in many committees
  - Alternatives that win a fixed sequence of pairwise comparisons (e.g., ((a vs. b) vs. c), etc.).

A Curious Preference Profile

33%	16%	3%	8%	18%	22%
a	*b*	*c*	*c*	*d*	*e*
b	*d*	*d*	*e*	*e*	*c*
c	*c*	*b*	*b*	*c*	*b*
d	*e*	*a*	*d*	*b*	*d*
e	*a*	*e*	*a*	*a*	*a*

Example due to Michel Balinski

- Plurality: ***a* wins**
- Borda: ***b* wins**
- Sequential majority comparisons (any order): ***c* wins**
- Instant-runoff: ***d* wins**
- Plurality with runoff: ***e* wins**

Rational Choice Theory

- A prerequisite for analyzing collective choice is to understand individual choice.
- Let U be a finite universe of alternatives.
- A *choice function* f maps a feasible set $A \subseteq U$ to a choice set $f(A) \subseteq A$.
  - We require that $f(A)=\emptyset$ only if $A=\emptyset$.
- Not every choice function complies with our intuitive understanding of rationality.
  - Certain patterns of choice from varying feasible sets may be deemed inconsistent, e.g., choosing a from $\{a,b,c\}$, but b from $\{a,b\}$.

A	$f(A)$
ab	a
bc	b
ac	a
abc	a

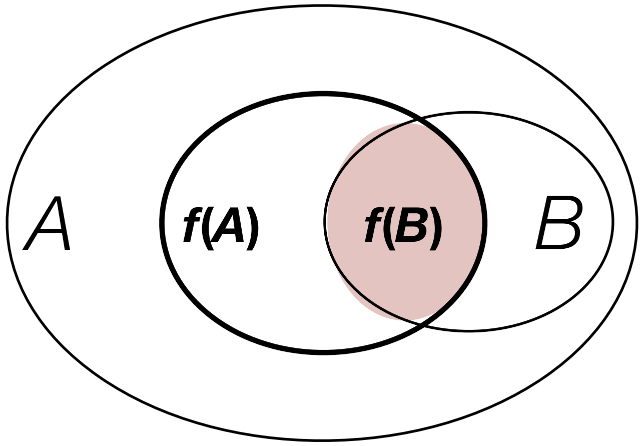
Rationalizable Choice

- Binary preference relation $\succcurlyeq$ on U
  - $x \succcurlyeq y$ is interpreted as *"x is at least as good as y".*
  - $\succcurlyeq$ is assumed to be transitive and complete.
- Best alternatives
  - For a binary relation $\succcurlyeq$ and a feasible set A,
    $Max(\succcurlyeq,A)= \{x \in A \mid \nexists y \in A \text{ such that } y \succ x\}$
- f is rationalizable if there exists a preference relation $\succcurlyeq$ on U such that $f(A)=Max(\succcurlyeq,A)$ for all A.
  - The previously mentioned choice function f with $f(\{a,b,c\})=\{a\}$ and $f(\{a,b\})=\{b\}$ cannot be rationalized.

Consistent Choice

- It would be a nice if the non-existence of a rationalizing relation could be pointed out by finding inconsistencies.
- f satisfies consistency if for all A,B with $B \subseteq A$,

  $f(A) \cap B \neq \emptyset$ implies $f(B) = f(A) \cap B$.



  - Consequence: If x is chosen from a feasible set, then it is also chosen from all subsets that contain x.
  - Example: Plurality does not satisfy consistency (when scores are computed for each feasible set).
    - $f(\{a,b,c\}) = \{a\}$ and $f(\{a,b\}) = \{b\}$

3	**2**	**2**
a	b	c
b	c	b
c	a	a

- Theorem (Arrow, 1959): A choice function is rationalizable iff it satisfies consistency.

From Choice to Social Choice

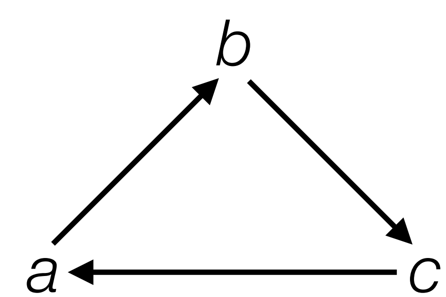
- *N* is a finite set of at least two voters.
- *R*(*U*) is the set of all preference relations over *U*.
- Every $R=(\succcurlyeq_1, \ldots, \succcurlyeq_{|N|}) \in R(U)^{|N|}$ is called a *preference profile*.
- A *social choice function (SCF)* is a function *f* that assigns a choice function to each preference profile.
  - An SCF is rationalizable (consistent) if its underlying choice functions are rationalizable (consistent) for all preference profiles.
  - We will write *f*(*R*,*A*) as a function of both *R* and *A*.
- Let $n_{xy} = |\{i \in N \mid x \succcurlyeq_i y\}|$ and define the *majority rule relation* as
  $(x \; R_M \; y) \Leftrightarrow n_{xy} > n_{yx}$.

Condorcet's Paradox

Marquis de Condorcet

- Social choice from feasible sets of size two is easy.
  - The *majority rule SCF* is defined as $f(R,\{x,y\}) = Max(R_M,\{x,y\})$.
  - Majority rule can easily be characterized using uncontroversial axioms (e.g., May, 1952).
- Problems arise whenever there are more than two alternatives.
  - Condorcet paradox (1785): R_M can be intransitive.
  - Alternative x is a *Condorcet winner* in A if $x\ R_M\ y$ for all $y \in A \setminus \{x\}$.
  - An SCF f is a *Condorcet extension* if $f(R,A)=\{x\}$ whenever x is a Condorcet winner in A.

1	1	1
a	b	c
b	c	a
c	a	b



Arrow's Impossibility


Kenneth J. Arrow

- An SCF satisfies *independence of infeasible alternatives (IIA)* if the choice set only depends on preferences over alternatives within the feasible set.
- An SCF satisfies *Pareto-optimality* if an alternative will not be chosen if there exists another alternative such that all voters prefer the latter to the former.
- An SCF is *dictatorial* if there exists a voter whose most preferred alternative is always uniquely chosen.
- Theorem (Arrow, 1951): Every rationalizable SCF that satisfies IIA and Pareto-optimality is dictatorial when $|U|\geq 3$.
  - Nipkow (2009) has verified a proof of Arrow's theorem using Isabelle.
  - Tang & Lin (2009) reduced the statement to a finite base case that was solved by a computer.

What now?

- Rationalizability (or, equivalently, consistency) is incompatible with collective choice when $|U|\geq 3$.
  - Dropping non-dictatorship is unacceptable.
  - Dropping Pareto-optimality offers little relief (Wilson, 1972).
  - Dropping IIA offers little relief (Banks, 1995).
- In this tutorial, we will consider two escape routes from Arrow's impossibility:
  - SCFs that satisfy weaker notions of consistency
    - Top cycle, uncovered set, Banks set, tournament equilibrium set
  - Randomized SCFs
    - Random dictatorship, maximal lotteries

What now?

- Two further escape routes (not considered in this tutorial)
  - Restricted domains of preferences
    - dichotomous preferences: approval voting
    - single-peaked preferences: median voting
  - Replace consistency with variable-electorate consistency
    - scoring rules
      - e.g., plurality, Borda
      - Smith and Young's characterization
    - Kemeny's rule
      - Young and Levenglick's characterization
      - computational intractability (NP-hard, even for four voters)

Interlude: Algorithms & Complexity

- One of the most important resources of an algorithm is time.
- An algorithm is called efficient if its running time is polynomial in its input size *n*.
  - Running time is bounded by n^k for constant k
- An essential question is whether a given computational problem admits an efficient algorithm.
  - If so, a natural follow-up task is to study and optimize the asymptotic and/or exact running time of this algorithm.
- Why *polynomial running time*?

Polynomial vs. Exponential Algorithms

[Garey & Johnson, 1979]

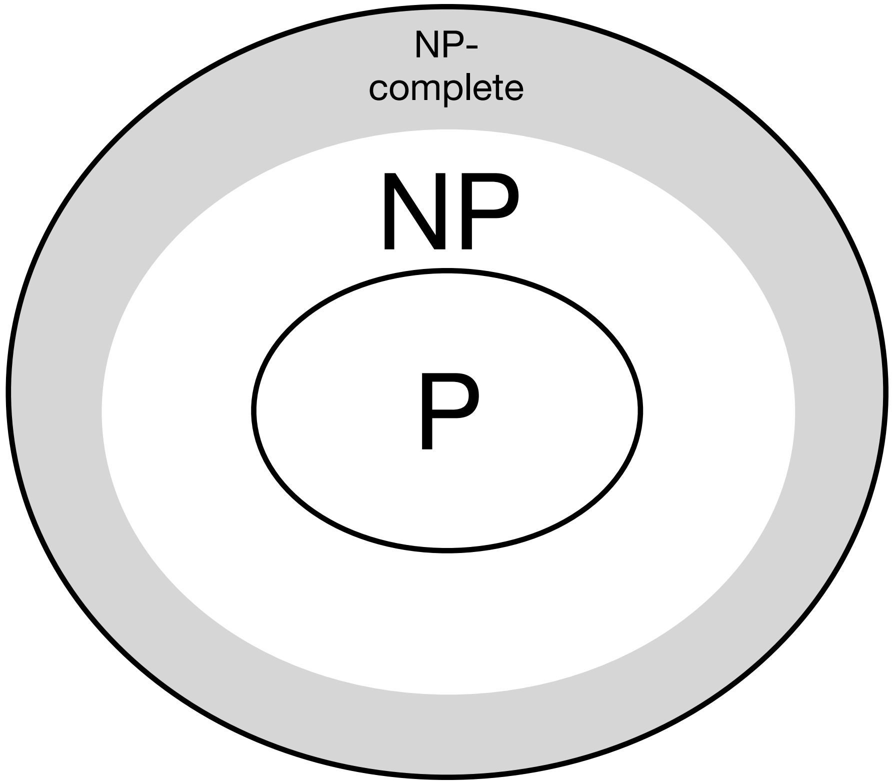
	10	20	30	40	50	60
n	0.00001 sec.	0.00002 sec.	0.00003 sec.	0.00004 sec.	0.00005 sec.	0.00006 sec.
n^2	0.0001 sec.	0.0004 sec.	0.0009 sec.	0.0016 sec.	0.0025 sec.	0.0036 sec.
n^3	0.001 sec.	0.008 sec.	0.027 sec.	0.064 sec.	0.125 sec.	0.216 sec.
n^5	1 sec.	3.2 sec.	24.3 sec.	1.7 min.	5.2 min.	13.0 min.
2^n	0.001 sec.	1.0 sec.	17.9 min.	12.7 days	35.7 years	366 centuries
3^n	0.059 sec.	58 min.	6.5 years	3855 centuries	$2 \cdot 10^8$ centuries	$1.3 \cdot 10^{13}$ centuries

Efficient Algorithms

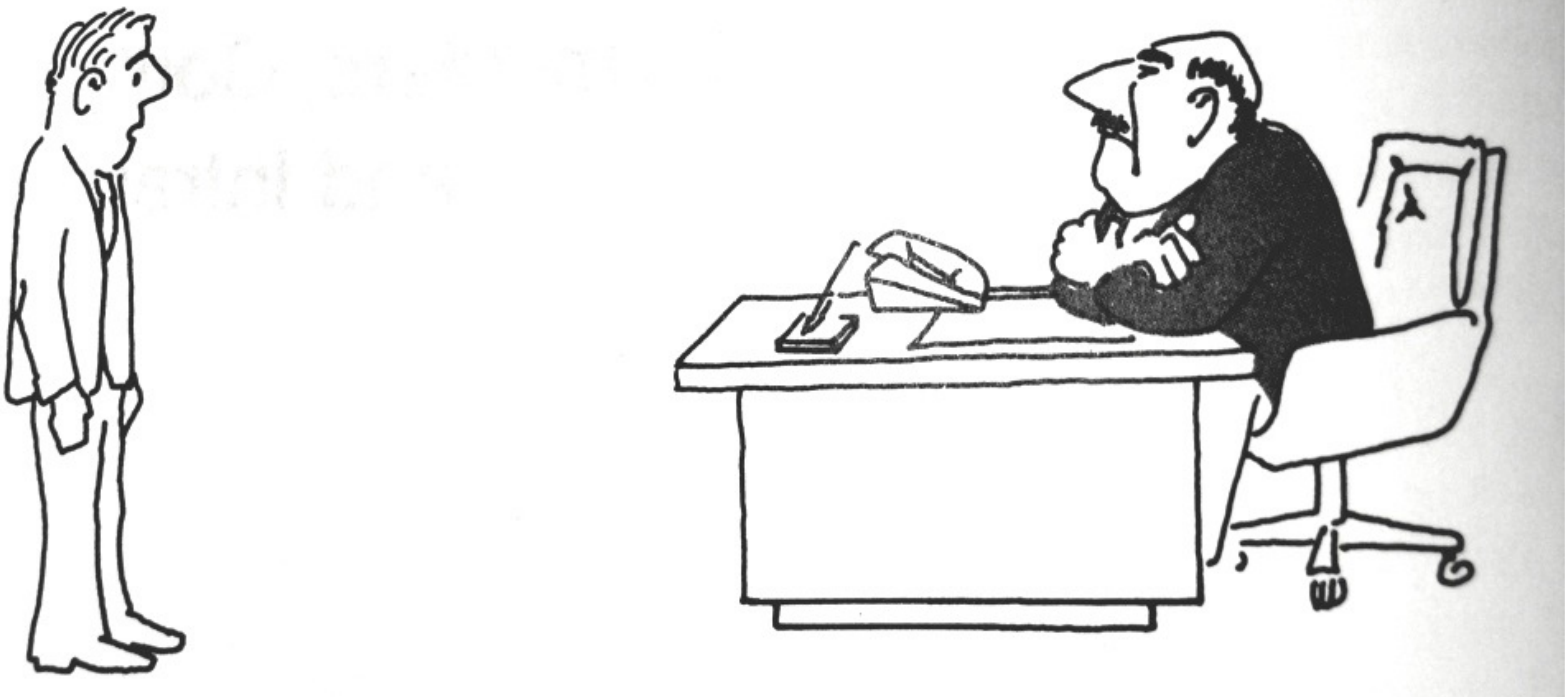
- How can we find an efficient algorithm?
  - Sometimes standard techniques are successful, e.g., greedy algorithms, divide and conquer, dynamic programming, linear programming, reduction to problems that can be solved efficiently.
  - Sometimes new insights into the structure of the problem at hand are required.
- How can we show that no efficient algorithm exists?
  - In almost all cases, we can't.
  - Frequently, we can prove something almost as powerful: NP-hardness.

P and NP

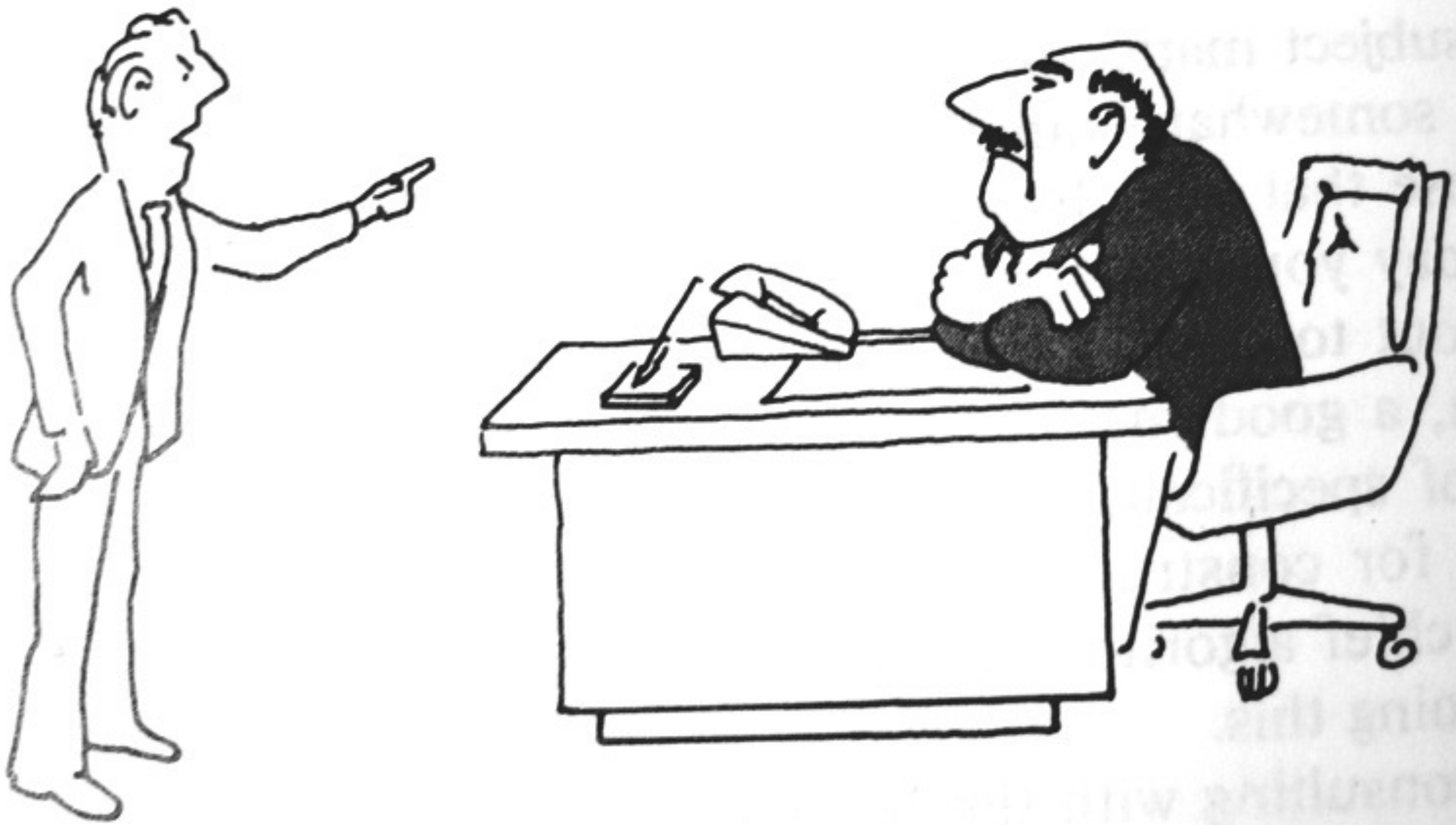
- P (problems that can be solved in polynomial time on a deterministic Turing machine)
  - Problems in P admit efficient algorithms.
- NP (problems that can be solved in polynomial-time on a non-deterministic Turing machine)
  - Solutions can be verified in polynomial time.
  - NP-hard problems
    - at least as hard as every problem in NP (with respect to polynomial-time reductions)
    - There are no efficient algorithms for NP-hard problems if P≠NP.
  - NP-complete problems (NP-hard and in NP)



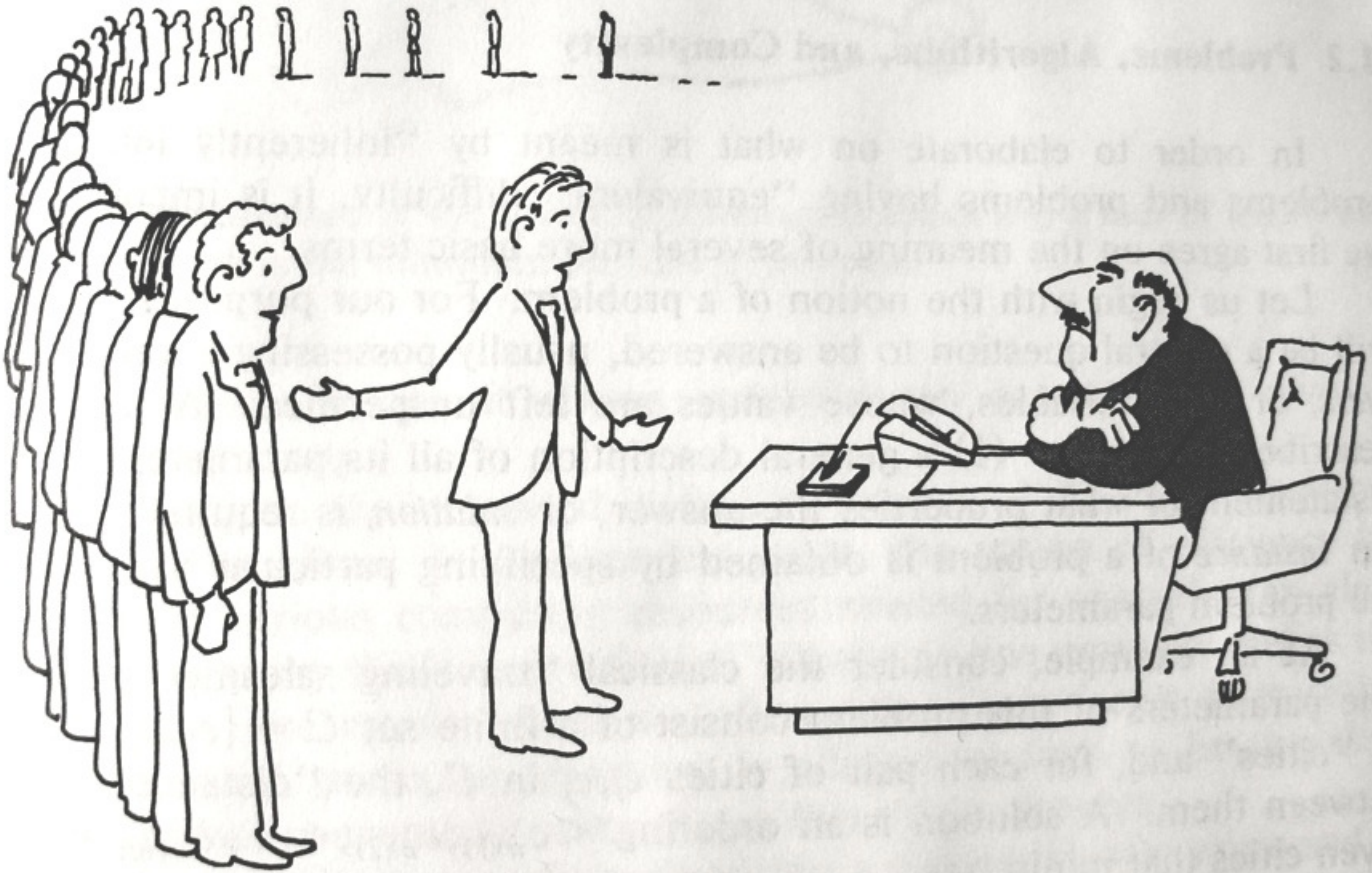
[Garey & Johnson, 1979]



"I can't find an efficient algorithm, I guess I'm just too dumb."



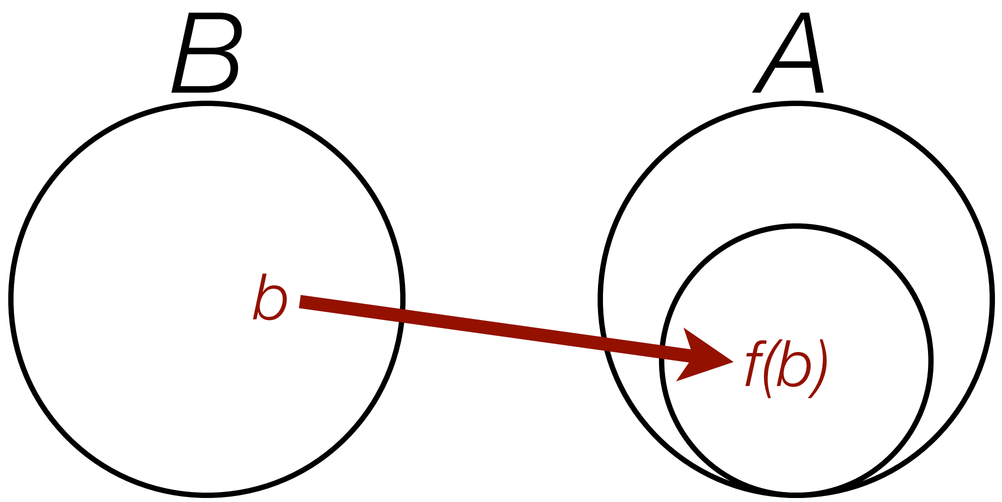
"I can't find an efficient algorithm, because no such algorithm is possible!"



"I can't find an efficient algorithm, but neither can all these famous people."

Hard and Easy Problems

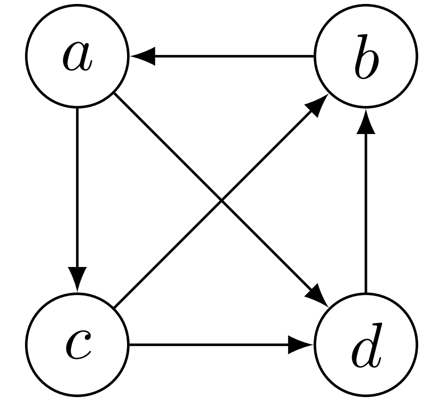
- How to show that problem *A* is
  - in P?
    - polynomial-time algorithm for *A*
  - NP-hard?
    - Reduction proof
    - Formulate *A* as a decision problem.
    - Choose an NP-hard decision problem *B*.
    - Construct an efficiently computable function *f*, which maps every instance *b* of *B* to an instance *f*(*b*) of *A* such that *f*(*b*) is *true* iff *b* is *true*.



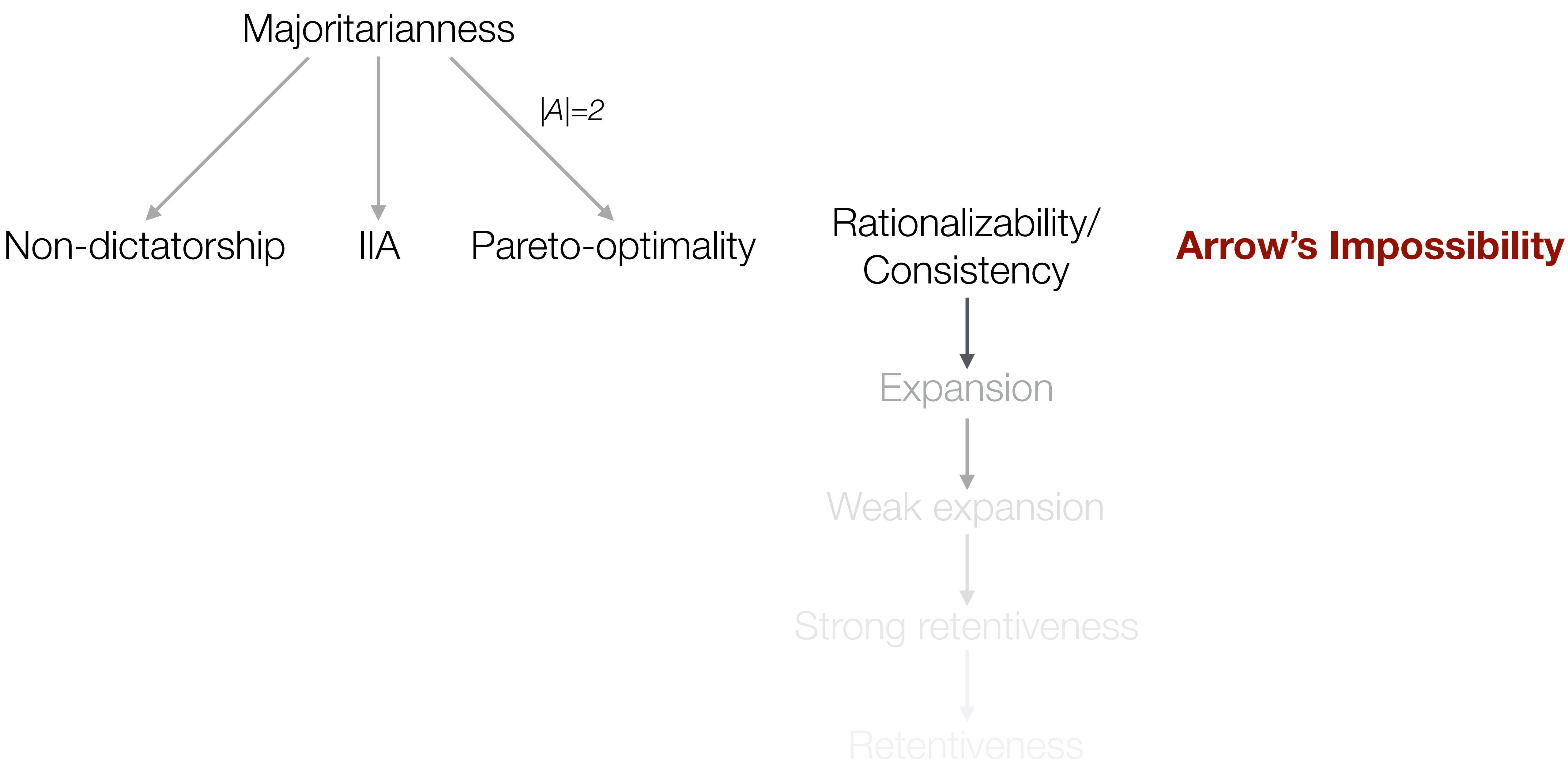
- SAT (Boolean satisfiability problem)
  - NP-complete, in particular in conjunctive normal form, even if every clause contains only three literals (3SAT).
    $(x_1 \vee \neg x_2 \vee x_3) \wedge (\neg x_1 \vee x_2 \vee \neg x_3) \wedge \ldots$
  - How to deal with NP-hard problems?
    - Restriction, Parametrization, Heuristics, Randomization, Approximation

Weakly Consistent SCFs

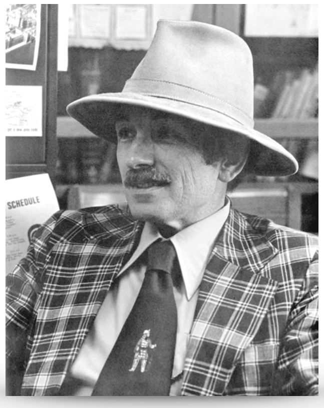
Tournaments



- For a given preference profile R, a feasible set A and majority rule R_M define a directed graph (A,R_M).
  - We say that b dominates a if $b \; R_M \; a$.
  - Every asymmetric directed graph is induced by some preference profile (McGarvey, 1953).
- A *majoritarian* SCF is an SCF whose output only depends on (A,R_M).
  - For simplicity, we will assume that individual preferences are anti-symmetric and that $|N|$ is odd. Hence, (A,R_M) is a tournament.
  - SCF f is said to be *finer* than SCF g if $f \subseteq g$.
  - Dominion $D(x)=\{y \in A \mid x \; R_M \; y\}$
  - Dominators $\overline{D}(x)=\{y \in A \mid y \; R_M \; x\}$


Majoritarianness
|A|=2
Non-dictatorship
IIA
Pareto-optimality
Rationalizability/
Consistency
Expansion
Weak expansion
Strong retentiveness
Retentiveness
Arrow’s Impossibility

The Top Cycle

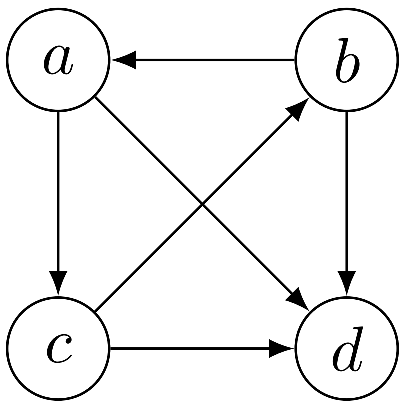

John I. Good

- Consistency can be weakened to expansion:
$B \subseteq A$ and $f(A) \cap B \neq \emptyset$ implies $f(B) \subseteq f(A)$.

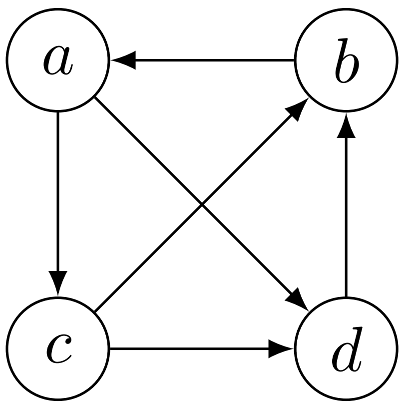
- Theorem (Bordes, 1976): There is a unique finest majoritarian SCF satisfying expansion: the top cycle.

- A *dominant set* is a nonempty set of alternatives $B \subseteq A$ such that for all $x \in B$ and $y \in A \setminus B$, $x\ R_M\ y$.
  - The set of dominant sets is totally ordered by set inclusion (Good, 1971).
  - Hence, every tournament contains a unique minimal dominant set called the top cycle (TC).
  - TC is a Condorcet extension.

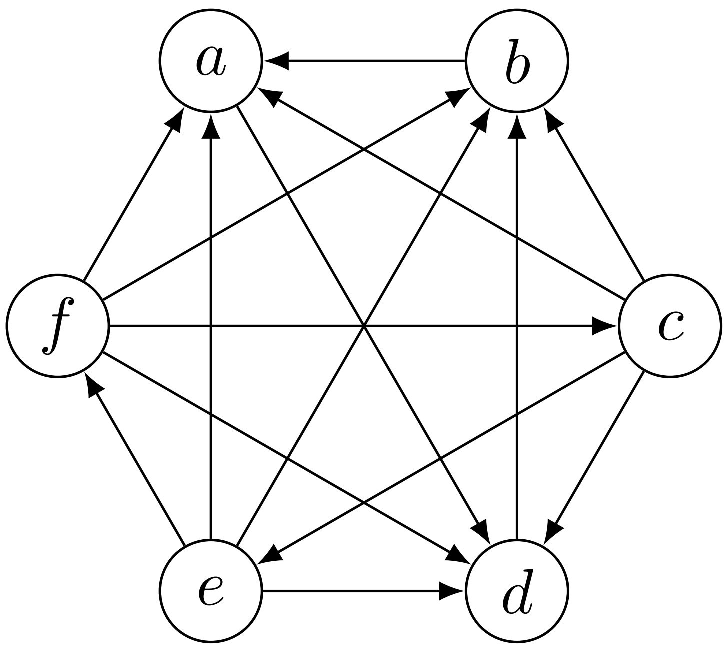
Examples



$TC(A,R_M)=\{a,b,c\}$



$TC(A,R_M)=\{a,b,c,d\}$



$TC(A,R_M)=\{c,e,f\}$

Transitive Closure

- The essence of Condorcet's paradox and Arrow's impossibility is that majority rule fails to be transitive.
  - Why not just take the transitive (reflexive) closure R_M^*?
- Theorem (Deb, 1977): $TC(A,R_M) = Max(R_M^*,A)$.
- Consequences
  - *TC* itself is a cycle. It is the source component in the directed acyclic graph of strongly connected components.
  - Linear-time algorithms for computing *TC* using Kosaraju's or Tarjan's algorithm for finding strongly connected components
    - Alternatively, one can initialize working set *B* with all alternatives of maximal outdegree and then iteratively add all alternatives that dominate an alternative in *B* until no more such alternatives can be found.

Top Cycle and Pareto-Optimality

- The top cycle is very large.
- In fact, it is so large that it fails to be Pareto-optimal when there are more than three alternatives (Ferejohn & Grether, 1977).

1	1	1
a	*b*	*d*
b	*c*	*a*
c	*d*	*b*
d	*a*	*c*

- Since Pareto-optimality is an essential ingredient of Arrow's impossibility, this escape route is (so far) not entirely convincing.
  - Although, technically, Arrow's theorem only requires Pareto-optimality for two-element sets (which the top cycle satisfies).

Majoritarianness

$|A|=2$

Non-dictatorship

IIA

Pareto-optimality

Rationalizability/
Consistency

Expansion

Weak expansion

Arrow's Impossibility

Top Cycle (TC)

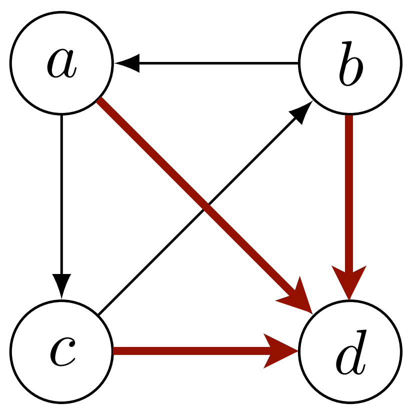
The Uncovered Set

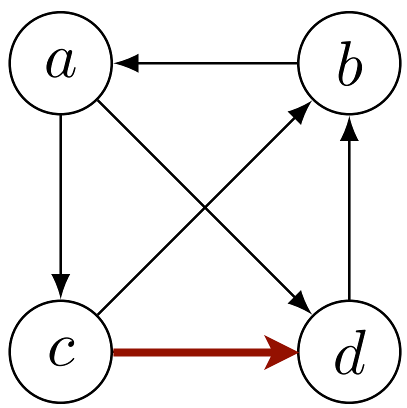
Peter C. Fishburn

Nicholas Miller

- Expansion can be further weakened to weak expansion: $f(A) \cap f(B) \subseteq f(A \cup B)$.
- Theorem (Moulin, 1986): There is a unique finest majoritarian SCF satisfying weak expansion: the uncovered set.
- Given a tournament (A,R_M), x covers y $(x\ C\ y)$, if $D(y) \subset D(x)$.
  - Proposed independently by Fishburn (1977) and Miller (1980)
  - Transitive subrelation of majority rule
  - The uncovered set (UC) consists of all uncovered alternatives, i.e., $UC(A,P_M) = Max(C,A)$.

Examples





$UC(A,R_M)=\{a,b,c\}$

$UC(A,R_M)=\{a,b,c\}$

$TC(A,R_M)=\{a,b,c,d\}$

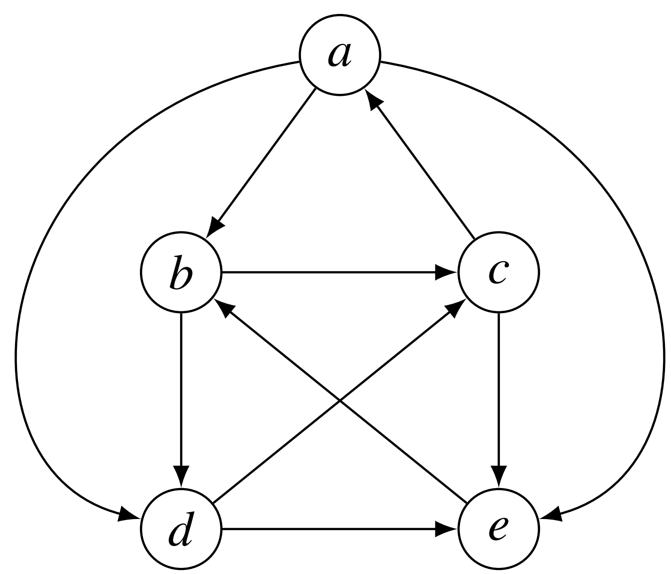
Properties of the Uncovered Set

- Since expansion ⇒ weak expansion, $UC \subseteq TC$.
  - *UC* is a Condorcet extension.
- *UC* satisfies Pareto-optimality.
  - Theorem (B. et al., 2016): *UC* is the largest majoritarian SCF satisfying Pareto-optimality.
- How can the uncovered set be efficiently computed?
  - Straightforward $O(n^3)$ algorithm that computes the covering relation for every pair of alternatives
  - Can we do better than that?

Uncovered Set Algorithm

- Equivalent characterization of *UC*
  - Theorem (Shepsle & Weingast, 1984): *UC* consists precisely of all alternatives that reach every other alternative in at most two steps.
    - Such alternatives are called kings in graph theory.
- Hence, *UC* can be computed by squaring the tournament's adjacency matrix.
  - Fastest known matrix multiplication algorithm (Le Gall, 2014): $O(n^{2.3728639})$
  - Just slightly faster than Vassilevska Williams, 2011: $O(n^{2.372873})$
  - Based on Coppersmith & Winograd (1990): $O(n^{2.376})$
  - Matrix multiplication is believed to be feasible in linear time ($O(n^2)$).

Uncovered Set Algorithm (Example)



$$\begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}$$

Majoritarianness

$|A|=2$

Non-dictatorship

IIA

Pareto-optimality

Rationalizability/
Consistency

Expansion

Weak expansion

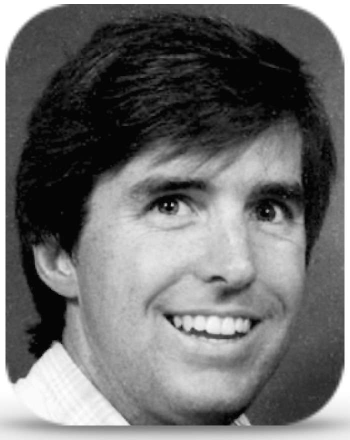
Strong retentiveness

Arrow's Impossibility

Top Cycle (TC)

Uncovered Set (UC)

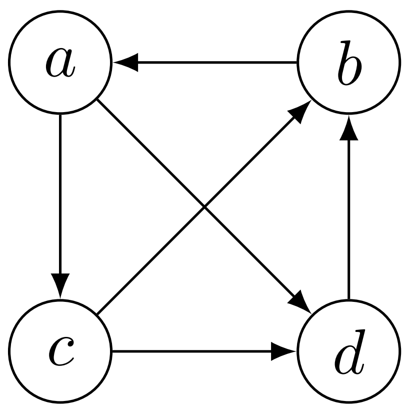
Banks Set



Jeffrey S. Banks

- Weak expansion can be weakened to strong retentiveness: $f(\overline{D}(x)) \subseteq f(A)$ for all $x \in A$.
- Theorem (B., 2011): There is a unique finest majoritarian SCF satisfying strong retentiveness: the Banks set.
- A *transitive subset* of a tournament (A,R_M) is a set of alternatives $B \subseteq A$ such that R_M is transitive within B.
- Let $Trans(A,R_M) = \{B \subseteq A \mid B \text{ is transitive}\}$.
- The Banks set (BA) consists of the maximal elements of all inclusion-maximal transitive subsets (Banks, 1985), i.e., $BA(A,R_M) = \{Max(R_M,B) \mid B \in Max(\supseteq, Trans(A,R_M))\}$

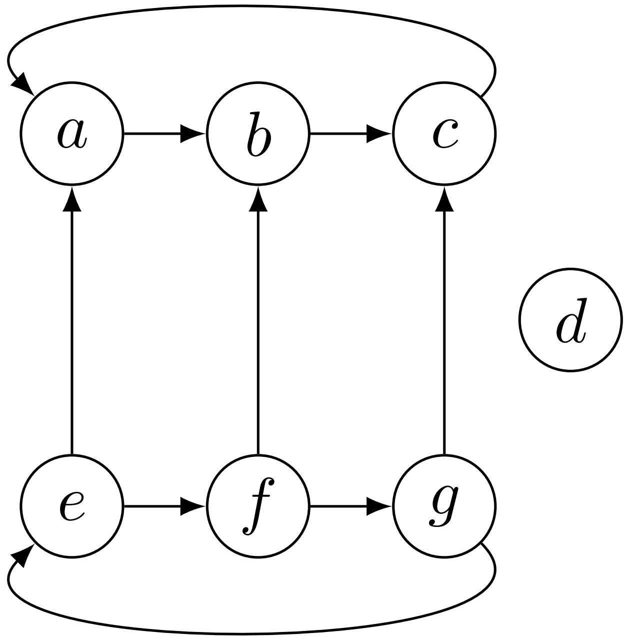
Examples



$UC(A,R_M)=\{a,b,c\}$

$BA(A,R_M)=\{a,b,c\}$

(All missing edges are pointing downwards.)



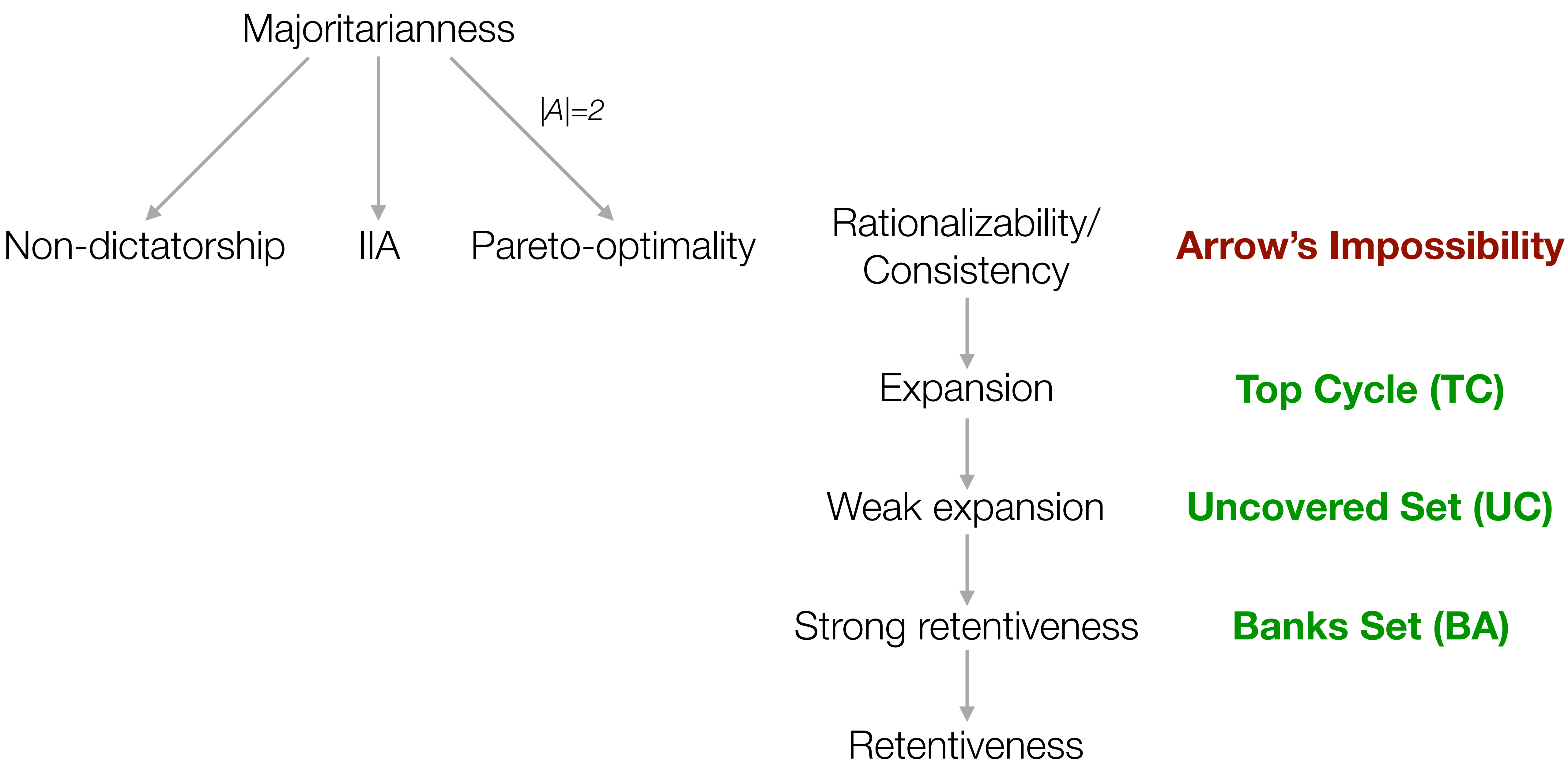
$TC(A,R_M)=\{a,b,c,d,e,f,g\}$

$UC(A,R_M)=\{a,b,c,d\}$

$BA(A,R_M)=\{a,b,c\}$

Properties of the Banks Set

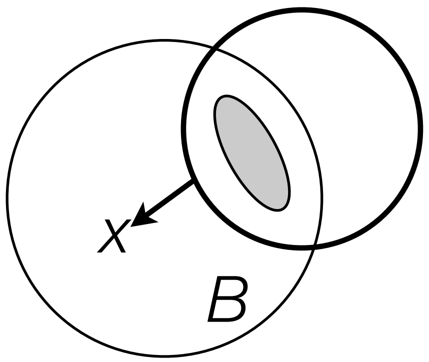
- Since expansion ⇒ weak expansion ⇒ strong retentiveness, $BA \subseteq UC \subseteq TC$.
  - As a consequence, BA is a Condorcet extension and satisfies Pareto-optimality.
- Random alternatives in BA can be found in linear time by iteratively constructing maximal transitive sets.
- Yet, computing the Banks set is NP-hard (Woeginger, 2003) and remains NP-hard even for 5 voters (Bachmeier et al., 2013).
- Strong retentiveness can be further weakened to retentiveness: $f(\overline{D}(x)) \subseteq f(A)$ for all $x \in f(A)$.


Majoritarianness
|A|=2
Non-dictatorship
IIA
Pareto-optimality
Rationalizability/
Consistency
Expansion
Weak expansion
Strong retentiveness
Retentiveness
Arrow's Impossibility
Top Cycle (TC)
Uncovered Set (UC)
Banks Set (BA)

Tournament Equilibrium Set

Thomas Schwartz

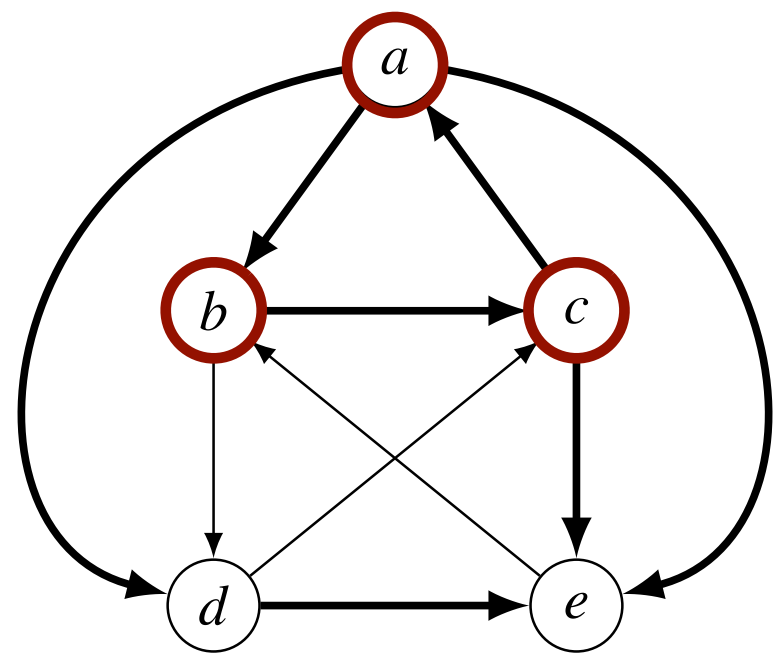
- Let f be an arbitrary choice function.
  - A non-empty set of alternatives B is *f-retentive* if $f(\overline{D}(x)) \subseteq B$ for all $x \in B$.
  - Idea: No alternative in the set should be "properly" dominated by an outside alternative.

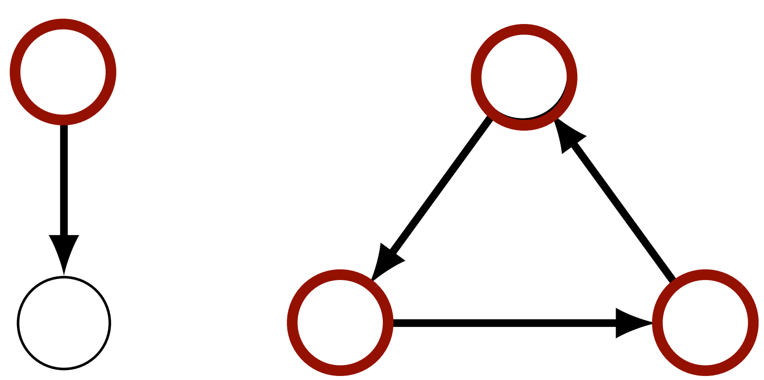


- $\mathring{f}$ is a new choice function that yields the union of all inclusion-minimal f-retentive sets.
  - $\mathring{f}$ satisfies retentiveness.
- The tournament equilibrium set (TEQ) of a tournament is defined as $TEQ = \mathring{TEQ}$.
  - Recursive definition (unique fixed point of ring-operator)
  - Theorem (Schwartz, 1990): $TEQ \subseteq BA$.

Example

- {a,b,c} is the unique minimal *TEQ*-retentive set.
  - $TEQ(\overline{D}(a)) = TEQ(\{c\}) = \{c\}$
  - $TEQ(\overline{D}(b)) = TEQ(\{a,e\}) = \{a\}$
  - $TEQ(\overline{D}(c)) = TEQ(\{b,d\}) = \{b\}$
  - $TEQ(\overline{D}(d)) = TEQ(\{a,b\}) = \{a\}$
  - $TEQ(\overline{D}(e)) = TEQ(\{a,c,d\}) = \{a,c,d\}$





A thick edge from y to x denotes that $y \in TEQ(\overline{D}(x))$.

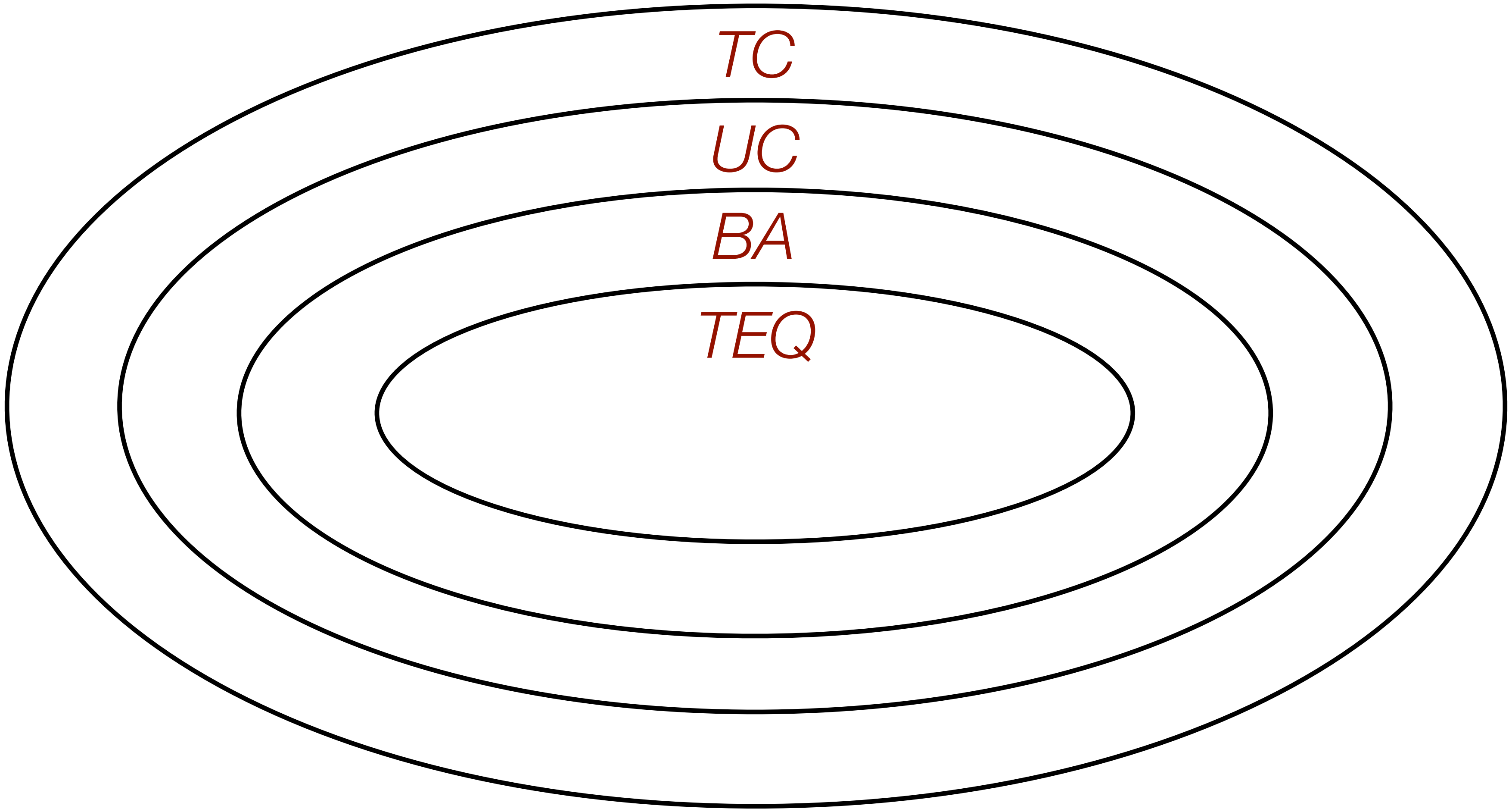
Properties of TEQ

- Computing *TEQ* is NP-hard (B. et al., 2010) and remains NP-hard even for 7 voters (Bachmeier et al., 2015).
  - The best known upper bound is PSPACE!
- Theorem (Laffond et al., 1993; Houy 2009; B., 2011; B. and Harrenstein, 2011): The following statements are equivalent:
  - ~~Every tournament contains a unique minimal *TEQ*-retentive set (Schwartz' Conjecture, 1990).~~
  - ~~*TEQ* is the unique finest majoritarian SCF satisfying retentiveness~~.
  - ~~*TEQ* satisfies monotonicity.~~
  - ~~*TEQ* satisfies independence of unchosen alternatives.~~
  - ~~*TEQ* is stable (and thus set-rationalizable)~~.
  - ~~*TEQ* is group-strategyproof (for Kelly's preference extension).~~
- Theorem (B., Chudnovsky, Kim, Liu, Norin, Scott, Seymour, and Thomassé, 2013): Schwartz's conjecture is false.

Properties of TEQ

- Theorem (B., Chudnovsky, Kim, Liu, Norin, Scott, Seymour, and Thomassé, 2013): Schwartz's conjecture is false.
  - non-constructive proof using the probabilistic method
  - neither a counter-example nor its size can be deduced from proof
  - smallest counter-example of this type requires ***10^{136}*** alternatives
- Schwartz's conjecture holds for ≤ 12 alternatives (B. et al., 2010).
- Schwartz's conjecture holds for ≤ 14 alternatives (Yang, 2016).
- Found no counter-example in extensive computer simulations
  - constructed counter-example with 24 alternatives (B. & Seedig, 2013)
- In principle, *TEQ* is severely flawed, but the existence of a counter-example seems to have no practical consequences whatsoever.
  - This casts doubt on the axiomatic method.

Weakly Consistent SCFs



Top Cycle (1971)	*TC*	expansion	$O(n^2)$
Uncovered Set (1977)	*UC*	weak expansion	$O(n^{2.38})$
Banks Set (1985)	*BA*	strong retentiveness	$2^{O(n)}$
Tournament Equilibrium Set (1990)	*TEQ*	(retentiveness)	$2^{O(n)}$

Probabilistic SCFs

Consistency and Lotteries

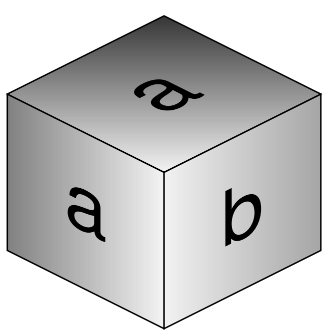
- Consistency for probabilistic SCFs can be defined as follows:
  - Let p be a lottery and A, B feasible sets such that p's support is contained in both A and B.
  - Then, p is chosen from A and from B iff it is chosen from $A \cup B$.
- This condition allows for attractive probabilistic SCFs, e.g.,
  - Random dictatorship (RD), and
  - Maximal lotteries (ML).

Probabilistic Social Choice

- Agents have complete and transitive preference relations $\succcurlyeq_i$ over a finite set of alternatives A.
- A *social decision scheme* f maps a preference profile $(\succcurlyeq_1, \ldots, \succcurlyeq_n)$ to a lottery $\Delta(A)$.

1	**1**	**1**
a	b	a
b	a	c
c	c	b

Random Dictatorship



$\frac{2}{3}\, a + \frac{1}{3}\, b$

- Special case: Random assignment (aka house allocation). A is the set of deterministic assignments.
  - Agents are indifferent between all assignments in which they are assigned the same object.

efficiency

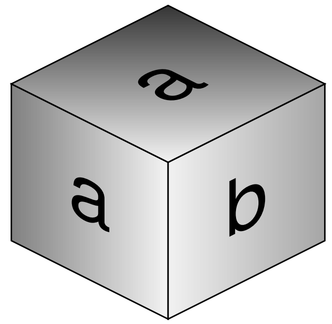
No agent can be made better off without making another one worse off

strategyproofness

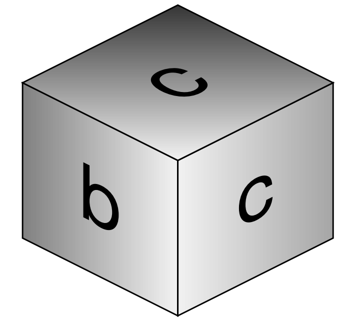
No agent can obtain a more preferred outcome by misreporting his preferences efficiency — strategyproofness

Only Dictatorship

strict preferences; Gibbard (1973), Satterthwaite (1975)



$\frac{2}{3}\, a + \frac{1}{3}\, b$



$\frac{1}{3}\, b + \frac{2}{3}\, c$

$a \succ b \succ c$

efficiency

there is no $p \in \Delta(A)$ such that

$p \succcurlyeq_i f(\cdot)$ for all $i \in N$ and

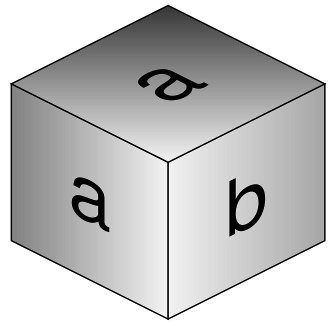
$p \succ_i f(\cdot)$ for some $i \in N$

strategyproofness

there is no $\succcurlyeq_i'$ such that

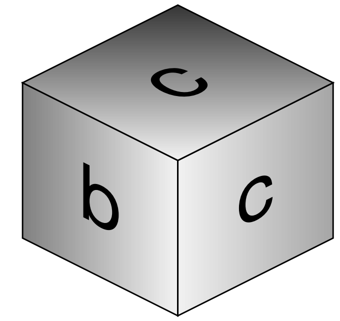
$f(\succcurlyeq_i', \cdot) \succ_i f(\succcurlyeq_i, \cdot)$

Extend preferences over alternatives to (incomplete) preferences over lotteries!



$\frac{2}{3}\, a + \frac{1}{3}\, b$

$a \succ b \succ c$



$\frac{1}{3}\, b + \frac{2}{3}\, c$

efficiency

there is no $p \in \Delta(A)$ such that

$p \succcurlyeq_i f(\cdot)$ for all $i \in N$ and

$p \succ_i f(\cdot)$ for some $i \in N$

strategyproofness

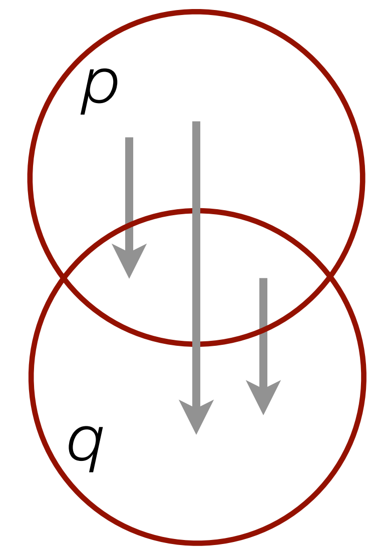
weak: there is no $\succcurlyeq_i'$ such that $f(\succcurlyeq_i', \cdot) \succ_i f(\succcurlyeq_i, \cdot)$

strong: for all $\succcurlyeq_i'$ it holds that $f(\succcurlyeq_i, \cdot) \succcurlyeq_i f(\succcurlyeq_i', \cdot)$

Extend preferences over alternatives to (incomplete) preferences over lotteries!

Sure Thing (ST)

	a	$\succ b$	$\succ c$	
$p=$ (	⅔	⅓	0	)
$q=$ (	0	⅓	⅔	)



- $p \succcurlyeq^{ST} q \quad \Leftrightarrow \quad \forall x \in \text{supp}(p) \setminus \text{supp}(q), y \in \text{supp}(q)\text{: } x \succ y$

  $\wedge \quad \forall x \in \text{supp}(p), y \in \text{supp}(q) \setminus \text{supp}(p)\text{: } x \succ y$

  $\wedge \quad \forall x \in \text{supp}(p) \cap \text{supp}(q)\text{: } p(x) = q(x)$

  - loosely based on Savage's sure-thing principle
  - inspired by non-probabilistic preference extensions due to Fishburn (1972) and Gärdenfors (1979)

Bilinear Dominance (BD)

	a	$\succ b$	$\succ c$
$p=$	(½	½	0)
$q=$	(⅓	⅓	⅓)

- $p \succcurlyeq^{BD} q \quad \Leftrightarrow \quad [\forall x,y \in A\colon x \succ y \Rightarrow p(x)\, q(y) \geq p(y)\, q(x)]$
  - for every pair of alternatives, it's more likely that *p* yields the better alternative and *q* the worse alternative
  - *p* is preferred to *q* for every consistent SSB utility function
  - Fishburn (1984), Aziz et al. (2015)

$$\forall \succcurlyeq\colon \succcurlyeq^{ST} \subseteq \succcurlyeq^{BD}$$

Stochastic Dominance (SD)

	a	$\succ b$	$\succ c$
$p=$ (	½	0	½)
$q=$ (	0	½	½)

- $p \succcurlyeq^{SD} q \quad \Leftrightarrow \quad \forall x \in A\colon \sum_{y \succcurlyeq x} p(y) \geq \sum_{y \succcurlyeq x} q(y)$
  - for every alternative, it's more likely that p yields something better
  - p yields more expected utility for every consistent vNM function
  - Bogomolnaia & Moulin (2001) and many others

$$\forall \succcurlyeq\colon \succcurlyeq^{ST} \subseteq \succcurlyeq^{BD} \subseteq \succcurlyeq^{SD}$$

Pairwise Comparison (PC)

	a	$\succ b$	$\succ c$
$p=$	(⅔	0	⅓)
$q=$	(0	1	0)

- $p \succcurlyeq^{PC} q \quad \Leftrightarrow \quad \forall x \in A: \sum_{x \succcurlyeq y} p(x)\, q(y) \geq \sum_{x \succcurlyeq y} q(x)\, p(y)$
  - it's more likely that p yields a better alternative
  - minimizes *ex ante* regret
  - $\succcurlyeq^{PC}$ is a complete relation for all $\succcurlyeq$
  - Blavatskyy (2006), Aziz et al. (2015)

$$\forall \succcurlyeq: \succcurlyeq^{ST} \subseteq \succcurlyeq^{BD} \subseteq \succcurlyeq^{SD} \subseteq \succcurlyeq^{PC}$$

efficiency

strategyproofness

weaker

strong *SD*-strategyproofness

PC-efficiency

PC-strategyproofness

SD-efficiency

SD-strategyproofness

ex post efficiency

BD-strategyproofness

ST-strategyproofness

Only Random Dictatorship

strict preferences; Gibbard (1977)

weaker

strong *SD*-strategyproofness

PC-efficiency

PC-strategyproofness

SD-efficiency

SD-strategyproofness

ex post efficiency

BD-strategyproofness

ST-strategyproofness

No assignment rule

strict preferences; equal treatment of equals; Bogomolnaia & Moulin (2001)

weaker

strong
SD-strategyproofness

PC-efficiency

PC-strategyproofness

SD-efficiency

SD-strategyproofness

ex post efficiency

BD-strategyproofness

ST-strategyproofness

Probabilistic Serial (PS) assignment rule
strict preferences, Bogomolnaia & Moulin (2001)

weaker

strong
SD-strategyproofness

PC-efficiency

PC-strategyproofness

SD-efficiency

SD-strategyproofness

ex post efficiency

BD-strategyproofness

ST-strategyproofness

Utilitarian rule (≈ approval voting/ maximal lotteries)
dichotomous preferences, Bogomolnaia & Moulin (2004)

weaker

strong
SD-strategyproofness

PC-efficiency — *PC*-strategyproofness

SD-efficiency

SD-strategyproofness

ex post efficiency

BD-strategyproofness

ST-strategyproofness

No anonymous and neutral social decision scheme
Aziz et al. (2014)

weaker

strong
SD-strategyproofness

PC-efficiency | *PC*-strategyproofness

SD-efficiency | *SD*-strategyproofness

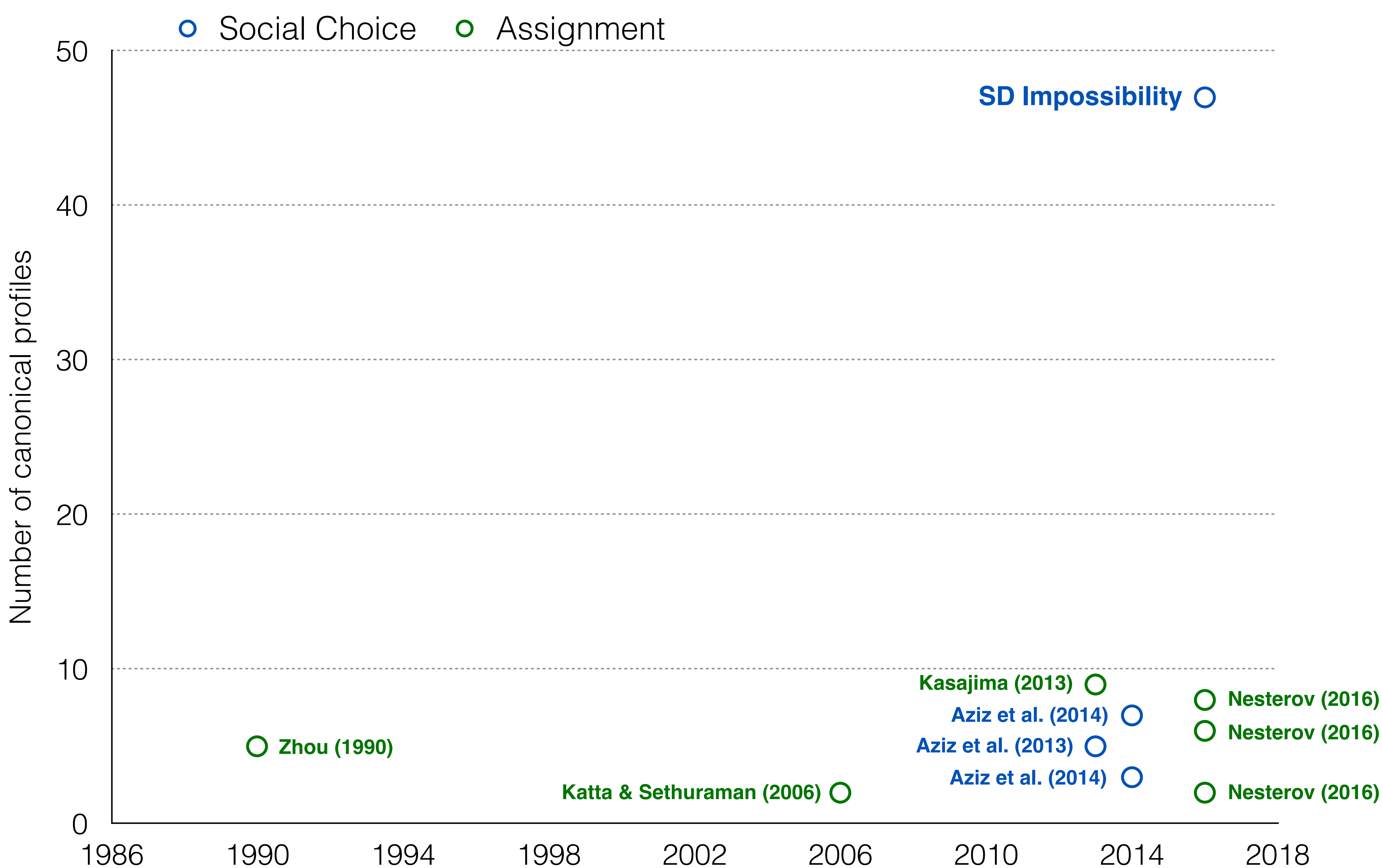
ex post efficiency | *BD*-strategyproofness

ST-strategyproofness

No anonymous and neutral social decision scheme
Brandl et al. (2016)

SD Impossibility

- requires at least 4 agents and at least 4 alternatives
  - more than 31 million possible preferences profiles
- was shown with the help of a computer (SMT solver)
- proof has been extracted from the solver’s output and brought into human-readable form
- operates on 47 canonical preference profiles and is very tedious to check
- has been verified by a computer (Isabelle/HOL)


Social Choice
Assignment
Number of canonical profiles
50
40
30
20
10
0
1986
1990
1994
1998
2002
2006
2010
2014
2018
SD Impossibility
Kasajima (2013)
Aziz et al. (2014)
Aziz et al. (2013)
Aziz et al. (2014)
Zhou (1990)
Katta & Sethuraman (2006)
Nesterov (2016)
Nesterov (2016)
Nesterov (2016)

weaker

strong
SD-strategyproofness

PC-efficiency | *PC*-strategyproofness

SD-efficiency — SD Impossibility — *SD*-strategyproofness

ex post efficiency | *BD*-strategyproofness

ST-strategyproofness

No anonymous and neutral social decision scheme
Brandl et al. (2016)

weaker

strong *SD*-strategyproofness

PC-efficiency

PC-strategyproofness

SD-efficiency

SD Impossibility

SD-strategyproofness

ex post efficiency

BD-strategyproofness

ST-strategyproofness

No pairwise social decision scheme

Aziz et al. (2014)

Random Serial Dictatorship

- Extension of random dictatorship to weak preferences
  - pick an ordering of agents uniformly at random
  - sequentially narrow down the set of alternatives by letting each agent restrict it to his most preferred ones.
- Widespread assignment rule (aka random priority)

1	1	1
a,*c*	*b*,*c*	*a*
b	*a*	*b*
		c

½ *a* + ⅙ *b* + ⅓ *c*

1,2,3: *c*
1,3,2: *a*
2,1,3: *c*
2,3,1: *b*
3,1,2: *a*
3,2,1: *a*

Random Serial Dictatorship

- *RSD* is strongly *SD*-strategyproof.
- *RSD* violates *SD*-efficiency.
  - first observed by Bogomolnaia & Moulin (2001) in assignment domain
  - $1/2\ a + 1/2\ b >_i^{SD} p$ for all $i \in N$.

1	**1**	**1**	**1**
a,c	*b,d*	*a*	*b*
b	*a*	*d*	*c*
d	*c*	*b,c*	*a,d*

$$p = 5/12\ a + 5/12\ b + 1/12\ c + 1/12\ d$$

- Computing *RSD* probabilities is #P-complete (Aziz et al., 2013).
  - Even checking whether the probability of a given alternative exceeds some fixed $\lambda \in (0,1)$ is NP-complete.

weaker

strong
SD-strategyproofness

PC-efficiency

PC-strategyproofness

SD-efficiency

Impossibility

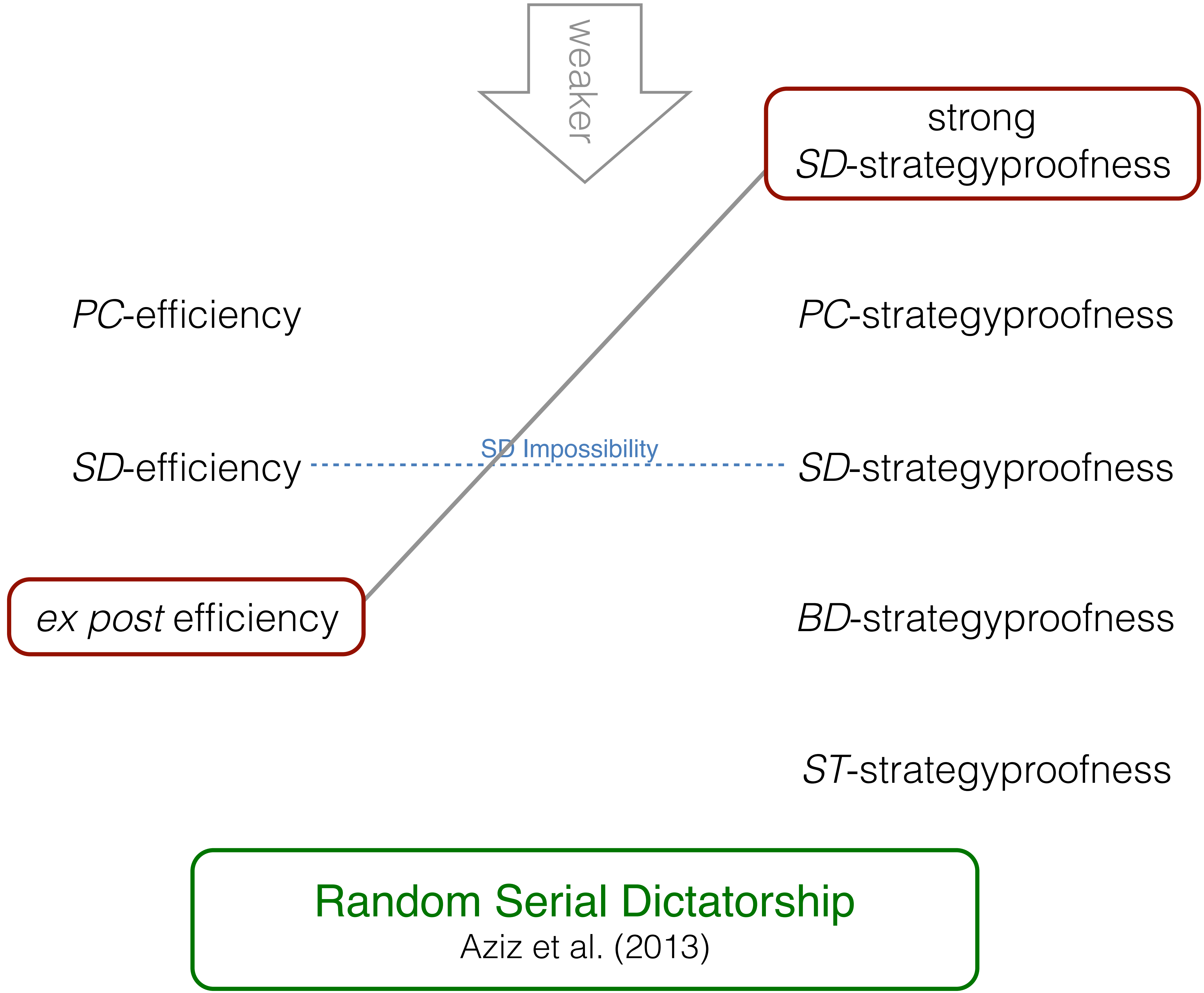
SD-strategyproofness

ex post efficiency

BD-strategyproofness

ST-strategyproofness

No pairwise social decision scheme
Aziz et al. (2014)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ST-strategyproofness
Random Serial Dictatorship
Aziz et al. (2013)

Maximal Lotteries

Germain Kreweras

Peter C. Fishburn

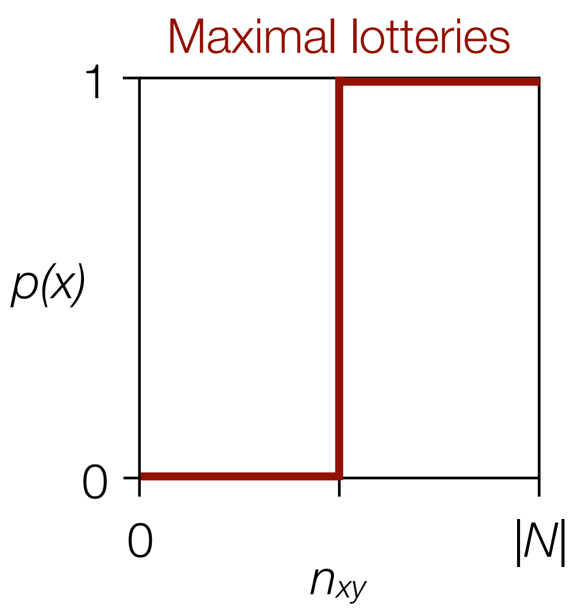
- Kreweras (1965) and Fishburn (1984)
  - Rediscovered by Laffond et al. (1993), Felsenthal and Machover (1992), Fisher and Ryan (1995), Rivest and Shen (2010)
- Let $g(x,y) = n_{xy} - n_{yx}$ be the *majority margin* of x and y.
- Alternative x is a (weak) Condorcet winner if $g(x,y) \geq 0$ for all y.
- Extend g to lotteries: $g(p,q) = \sum_{x,y} p(x) \cdot q(y) \cdot g(x,y)$
  - Expected majority margin
- p is a maximal lottery if $g(p,q) \geq 0$ for all q.
  - Randomized (weak) Condorcet winner
  - Always exists due to Minimax Theorem (v. Neumann, 1928)

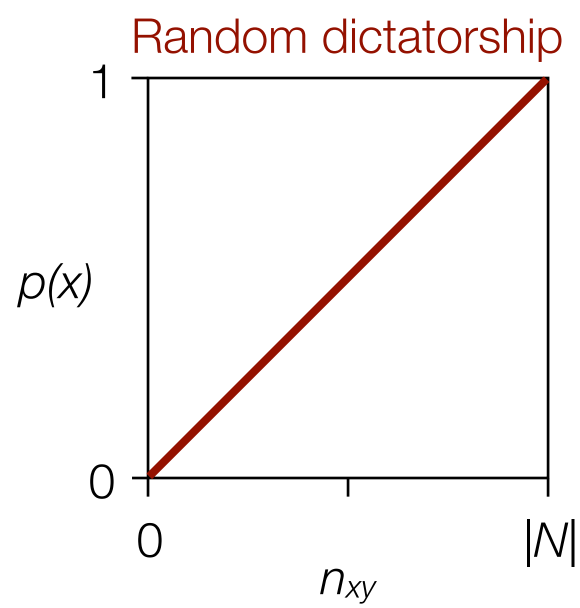
Germain Kreweras

Maximal Lotteries

Peter C. Fishburn

- Two alternatives





- *g* can be interpreted as a symmetric zero-sum game.
  - Maximal lotteries are mixed minimax strategies (or Nash equilibria).

2	**2**	**1**
a	*b*	*c*
b	*c*	*a*
c	*a*	*b*

$\frac{3}{5}\, a + \frac{1}{5}\, b + \frac{1}{5}\, c$

	a	*b*	*c*
a	0	1	-1
b	-1	0	3
c	1	-3	0

Germain Kreweras

Maximal Lotteries

Peter C. Fishburn

- Maximal lotteries are almost always unique.
  - Always unique for odd number of voters (Laffond et al., 1997)
- *ML* does not require asymmetry, completeness, or even transitivity of preferences.
- *ML* can be efficiently computed via linear programming.
- In the assignment domain, maximal lotteries are known as popular mixed matchings (Kavitha et al., 2011).
- *ML* is *PC*-efficient.

Germain Kreweras

Maximal Lotteries

Peter C. Fishburn

- *ML* violates *BD*-strategyproofness.

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & -1 \\ b & -1 & 0 & 1 \\ c & 1 & -1 & 0 \end{array}$$

1	**1**	**2**	**1**
a	*a*	*b*	*c*
c	*b*	*c*	*a*
b	*c*	*a*	*b*

R

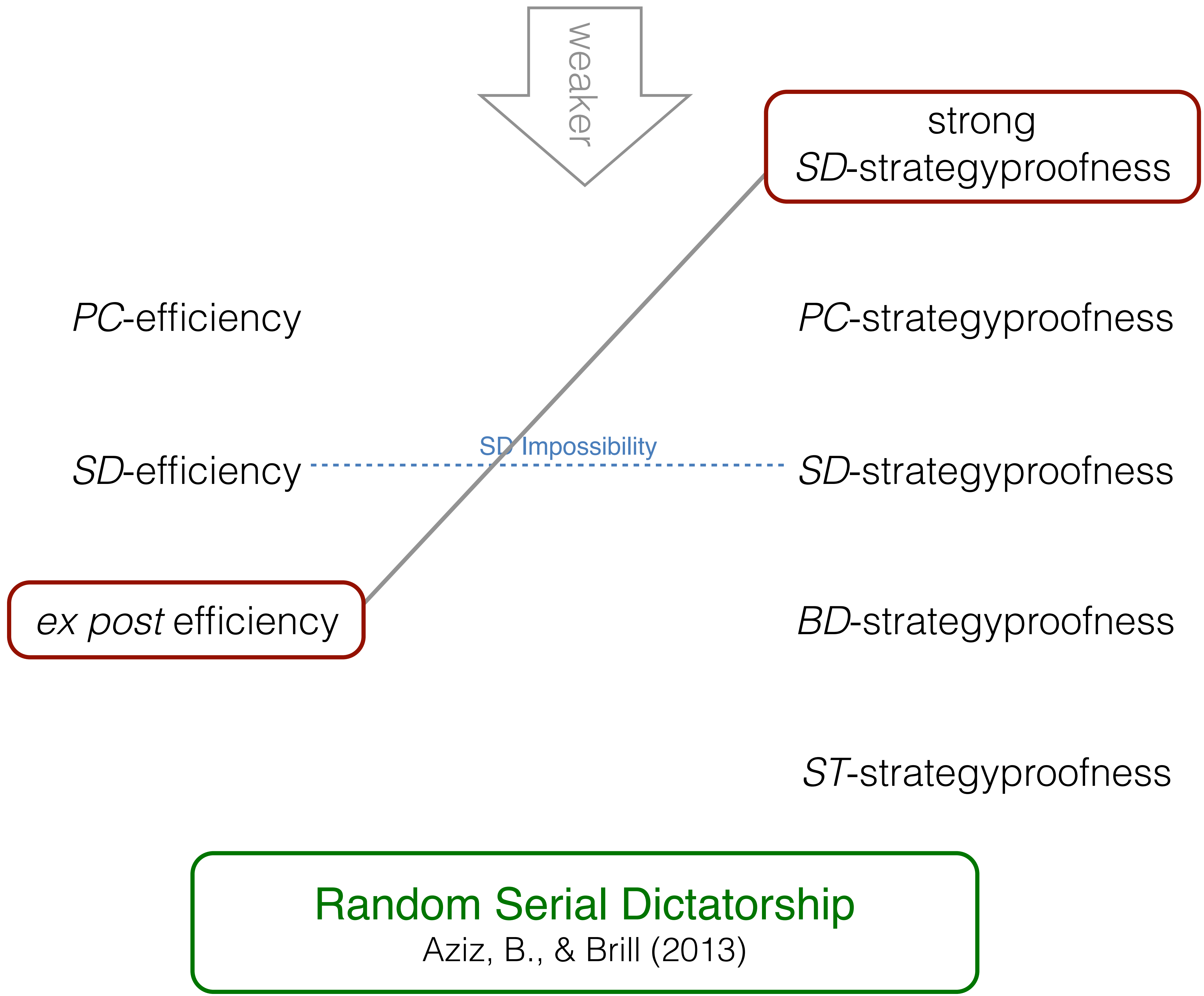
1	**1**	**2**	**1**
a	*a*	*b*	*c*
b	*b*	*c*	*a*
c	*c*	*a*	*b*

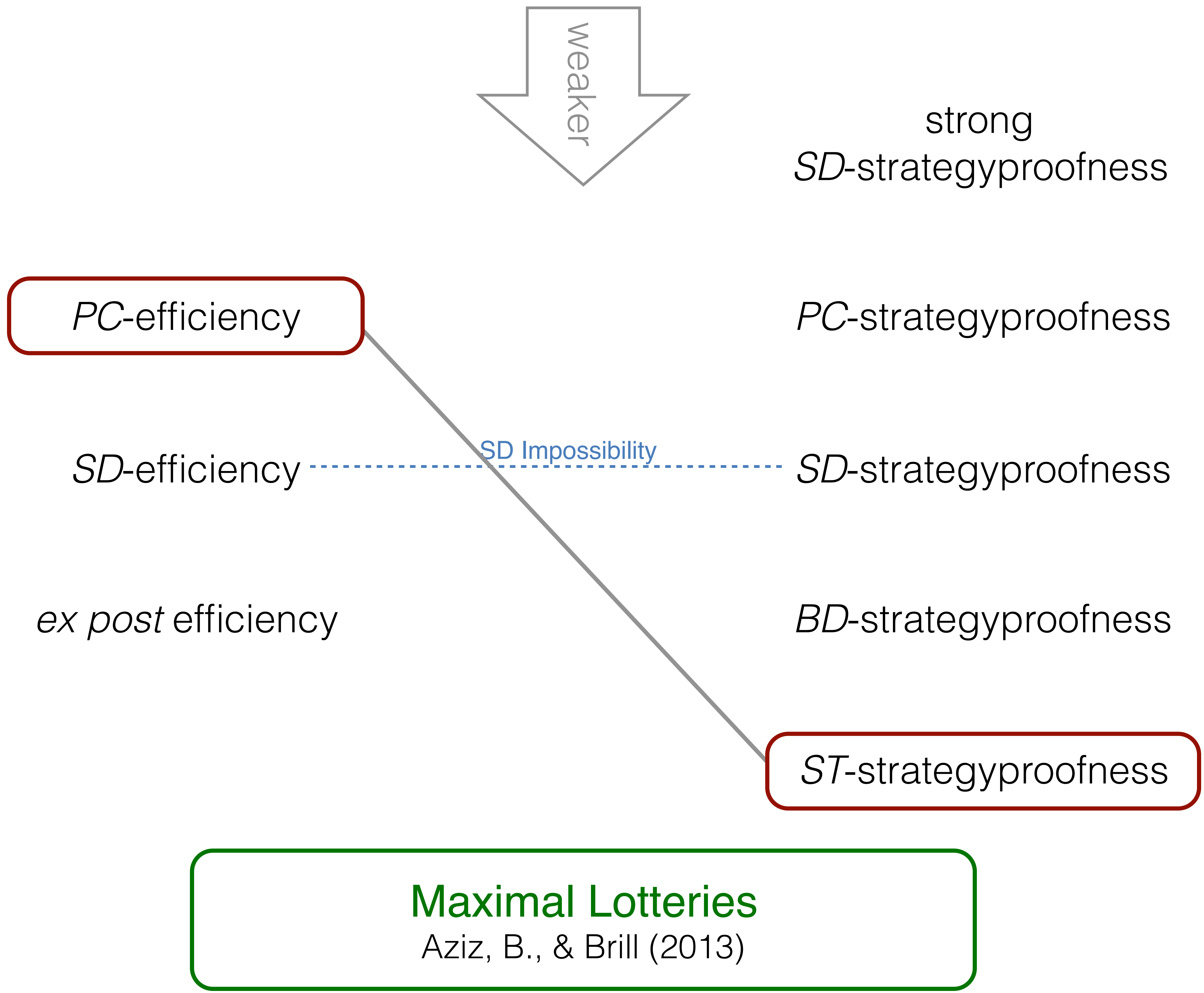
R'

$$\begin{array}{c|ccc} & a & b & c \\ \hline a & 0 & 1 & -1 \\ b & -1 & 0 & 3 \\ c & 1 & -3 & 0 \end{array}$$

$p = \frac{1}{3}\,a + \frac{1}{3}\,b + \frac{1}{3}\,c$ $\qquad$ $q = \frac{3}{5}\,a + \frac{1}{5}\,b + \frac{1}{5}\,c$

- *ML* satisfies *ST*-strategyproofness.


weaker
strong SD-strategyproofness
PC-efficiency
PC-strategyproofness
SD Impossibility
SD-efficiency
SD-strategyproofness
ex post efficiency
BD-strategyproofness
ST-strategyproofness
Random Serial Dictatorship
Aziz, B., & Brill (2013)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Maximal Lotteries
Aziz, B., & Brill (2013)

Intermediate Summary

- No social decision scheme satisfies moderate degrees of efficiency and strategyproofness.
- *RSD* is very strategyproof, but only a little efficient.
- *ML* is very efficient, but only a little strategyproof.
- Further results
  - *RSD* and *ML* are *ST*-group-strategyproof, but not *SD*-group-strategyproof.
  - No anonymous and neutral social decision scheme is *ex post* efficient and *BD*-group-strategyproof, even when preferences are dichotomous.
- *ML* can be characterized using consistency conditions.

Population-Consistency

> Whenever two disjoint electorates agree on a lottery, this lottery should also be chosen by the union of both electorates.

1	**1**
a	*b*
b	*c*
c	*a*

R

½ *a* + ½ *b*

1	**1**
a	*b*
c	*c*
b	*a*

S

½ *a* + ½ *b*

1	**1**	**2**
a	*a*	*b*
b	*c*	*c*
c	*b*	*a*

$R \cup S$

½ *a* + ½ *b*

- first proposed by Smith (1973), Young (1974), Fine & Fine (1974)
- also known as “reinforcement” (Moulin, 1988)
- famously used for the characterization of scoring rules and Kemeny

Agenda-Consistency

A lottery should be chosen from two agendas
iff it is also chosen in the union of both agendas.

1	1
a	b
d	c
b	d
c	a

R

$\frac{1}{2}\,a + \frac{1}{2}\,b$

1	1
a	b
b	c
c	a

$R|_A$

$\frac{1}{2}\,a + \frac{1}{2}\,b$

1	1
a	b
d	d
b	a

$R|_B$

$\frac{1}{2}\,a + \frac{1}{2}\,b$

$A=\{a,b,c\}$
$B=\{a,b,d\}$

- Sen (1971)'s α (contraction) and γ (expansion)
- at the heart of numerous impossibilities (e.g., Blair et al., 1976; Sen, 1977; Kelly, 1978; Schwartz, 1986)

Composition-Consistency

Composed preference profiles are treated component-wise.

In particular, alternatives are not affected by the cloning of other alternatives.

1	**1**	**1**	**1**
a	a	b	b'
b'	b	b'	b
b	b'	a	a

R

$\frac{1}{2}\, a + \frac{1}{4}\, b + \frac{1}{4}\, b'$

2	**2**
a	b
b	a

$R|_A$

$\frac{1}{2}\, a + \frac{1}{2}\, b$

2	**2**
b'	b
b	b'

$R|_B$

$\frac{1}{2}\, b + \frac{1}{2}\, b'$

$A=\{a,b\}$

$B=\{b,b'\}$

- Laffond, Laslier, and Le Breton (1996)
- cloning consistency precursors: Arrow and Hurwicz (1972), Maskin (1979), Moulin (1986), Tideman (1987)

Chevalier de Borda

Non-Probabilistic Social Choice

Marquis de Condorcet

- All scoring rules satisfy population-consistency. (Smith 1973; Young, 1974)
- No Condorcet extension satisfies population-consistency. (Young and Levenglick, 1978)
- Many Condorcet extensions satisfy composition-consistency. (Laffond et al., 1996)
- No Pareto-optimal scoring rule satisfies composition-consistency. (Laslier, 1996)
- Population-consistency and composition-consistency are incompatible in non-probabilistic social choice. (Brandl et al., 2016)
- *ML* is the only probabilistic SCF that satisfies population-consistency and composition-consistency. (Brandl et al., 2016)

Participation

No agent can obtain a more preferred lottery
by abstaining from an election.

	a	b	c
a	0	1	-1
b	-1	0	1
c	1	-1	0

1	**1**	**2**	**1**
a	a	b	c
c	b	c	a
b	c	a	b

R

⅓ a + ⅓ b + ⅓ c

1	**2**	**1**
a	b	c
b	c	a
c	a	b

R'

b

	a	b	c
a	0	0	-2
b	0	0	2
c	2	-2	0

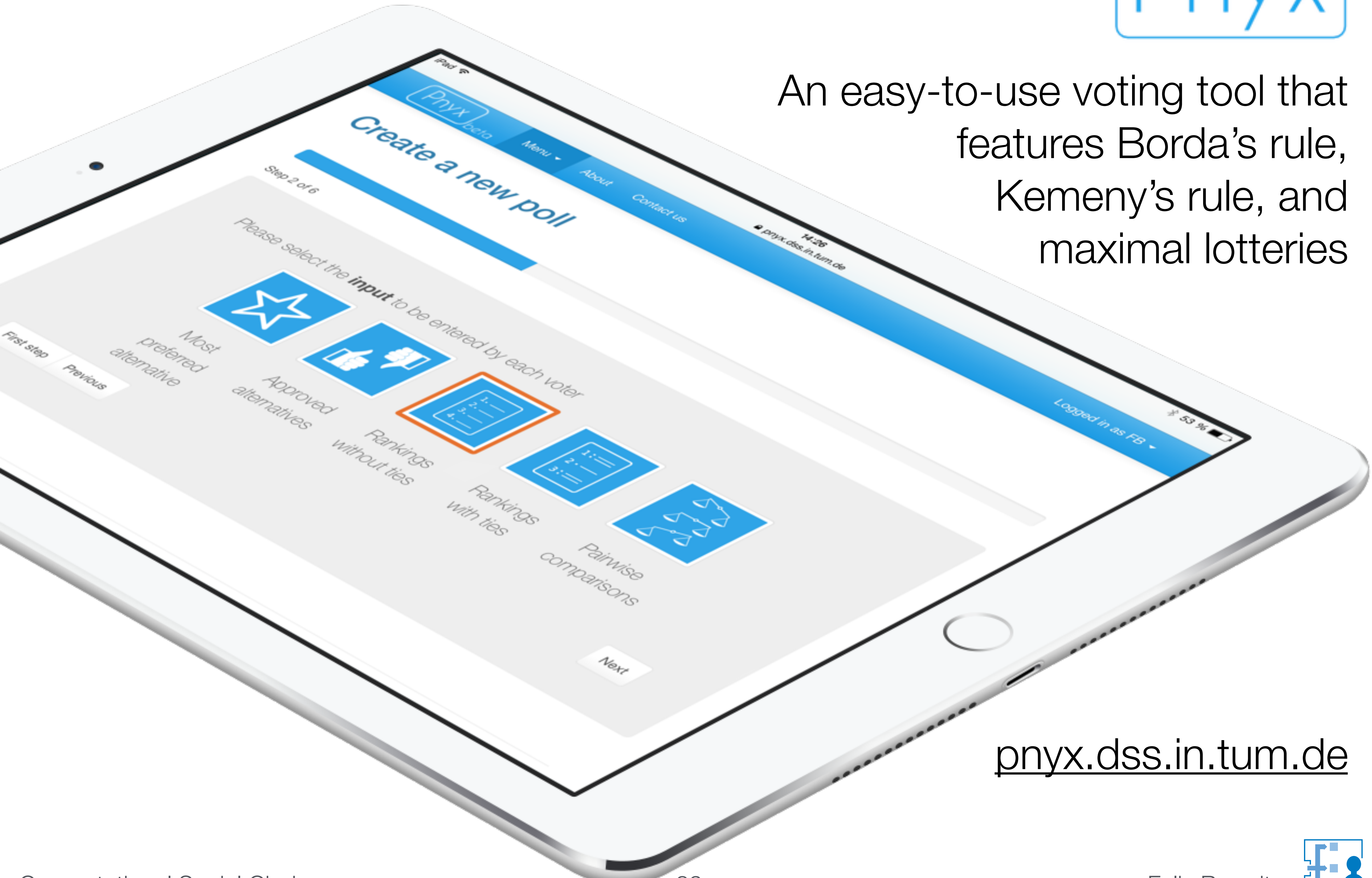
- cannot be satisfied by *resolute* Condorcet extensions (Moulin, 1988)
- satisfied by maximal lotteries with respect to the PC extension

	Maximal Lotteries	Random Serial Dictatorship	Borda's Rule
population-consistency	✅	only for strict prefs	✅
agenda-consistency	✅	✅	—
cloning-consistency	✅ even composition-consistency	✅	—
Condorcet-consistency	✅	—	—
(SD-) **strategyproofness**	—	✅ even strongly	—
(ST-) **group-strategyproofness**	✅	✅	—
(SD-) **participation**	✅ even PC-group-participation	✅ even very strongly	✅
(SD-) **efficiency**	✅	only for strict prefs otherwise only *ex post*	✅
efficient computability	✅	#P-complete in P for strict prefs	✅

Recommended Literature

- Allingham: *Choice Theory - A very short introduction*. Oxford University Press, 2002
- Austen-Smith and Banks: *Positive Political Theory I*, University of Michigan Press, 1999
- B., Conitzer, Endriss, Lang, and Procaccia: *Handbook of Computational Social Choice*. Cambridge University Press, 2016.
- Gärtner: *A Primer in Social Choice Theory*, Oxford University Press, 2009
- Moulin: *Axioms of Cooperative Decision Making*. Cambridge University Press, 1988
- Nitzan: *Collective Choice and Preference*. Cambridge University Press, 2010

Pnyx



An easy-to-use voting tool that features Borda's rule, Kemeny's rule, and maximal lotteries

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