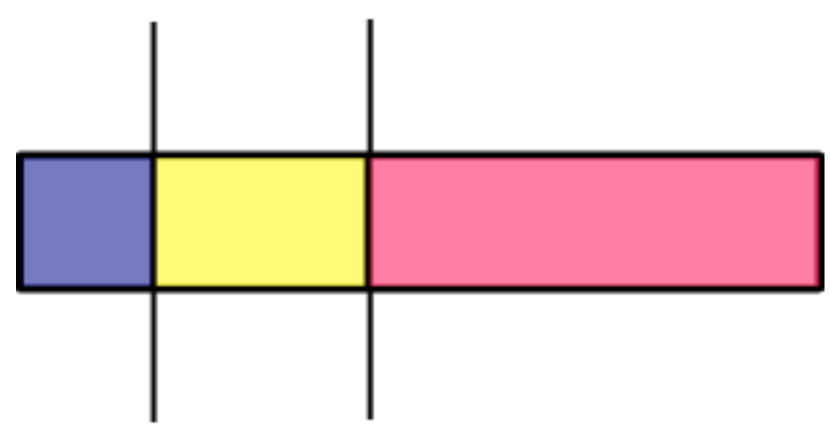
# Mathematical Aspects of Fair Division

HSE Summer School

Francis Edward Su
(Harvey Mudd College)



# How to cut a cake fairly?

- Math makes these words precise:
  "cake" "cut" "fair" "how"
- Math asks:
  existence? construction? properties?

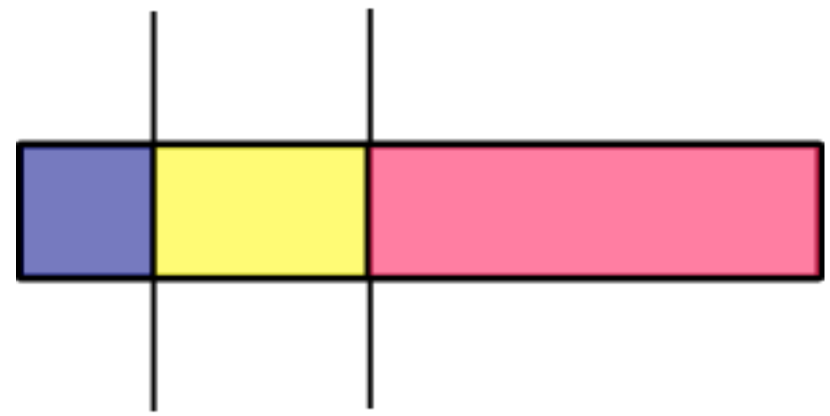
# “Cake”

- Properties
  - Desirable? Undesirable?
  - Utility, Shape
  - Divisible? Indivisible?

# "Fair"

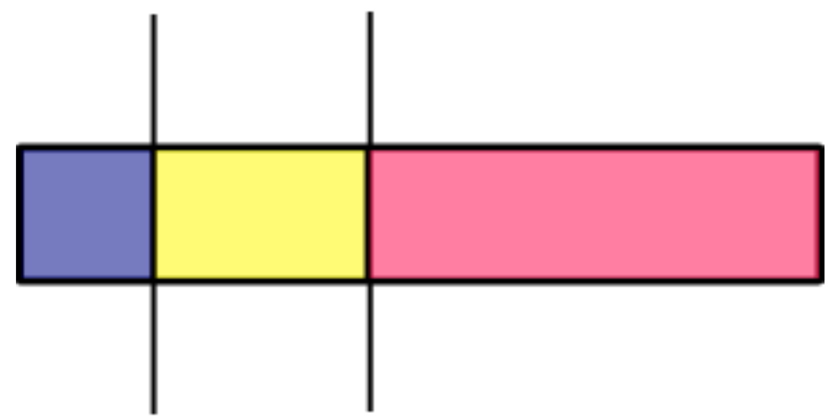
- PROPORTIONAL: each thinks her piece is at least 1/N in her measure
- ENVY-FREE: each thinks her piece is biggest
- EQUITABLE: each views his piece as the same as everyone else views their piece.
- EFFICIENT: there is no other division that dominates this one for ALL players.

# "How?"



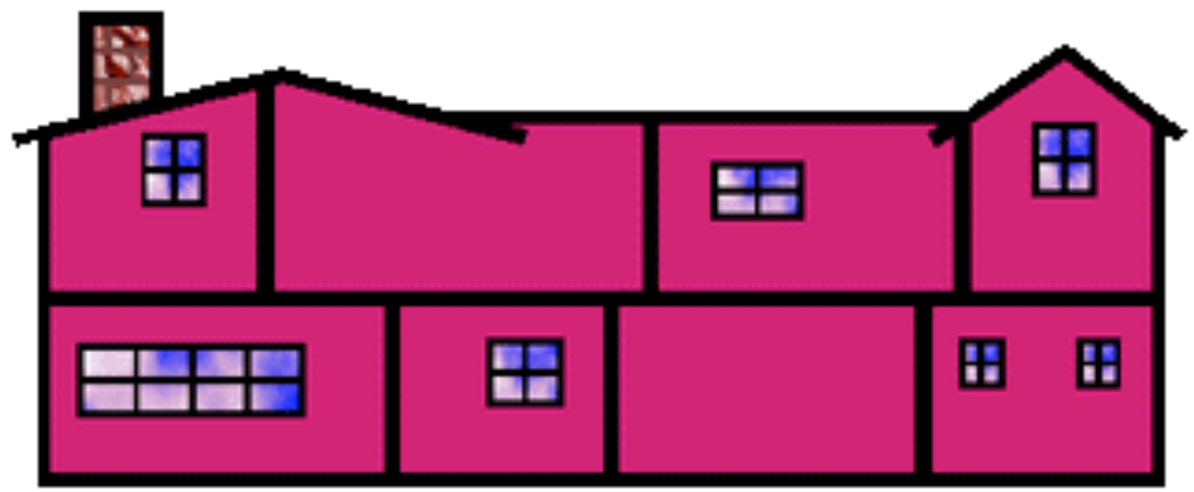
- Does a solution exist?
- Are there procedures for finding a "fair" division?

# “Cut”



- What kinds of divisions possible?

# Rental Harmony



- Brad and friends
- Is there always way to split rent fairly? (so each will choose a different room) How?

# Cake

- Cake division: infinitely divisible good, assume players have countably additive measures: $\mu_1, \mu_2, \ldots \mu_N$ with no *atoms*.
- Parallel cuts (cake linear)
- Not pie, eg Barbanel-Brams-Stromquist(2009)

# Existence

- Suppose a cake division is $\{A_1, A_2, \ldots, A_N\}$.
- Neyman(1946): there's a division where $\mu_i(A_j) = 1/N$ for all $i$ and $j$
- "Perfect" fairness

# Existence

- Vector measure: given set A,
  consider vector <$\mu_1(A), \mu_2(A), \ldots, \mu_N(A)$>
- Lyapunov(1940): range of non-atomic vector measure is compact and convex.
- Consequence:
  N people can agree on *a:b* split

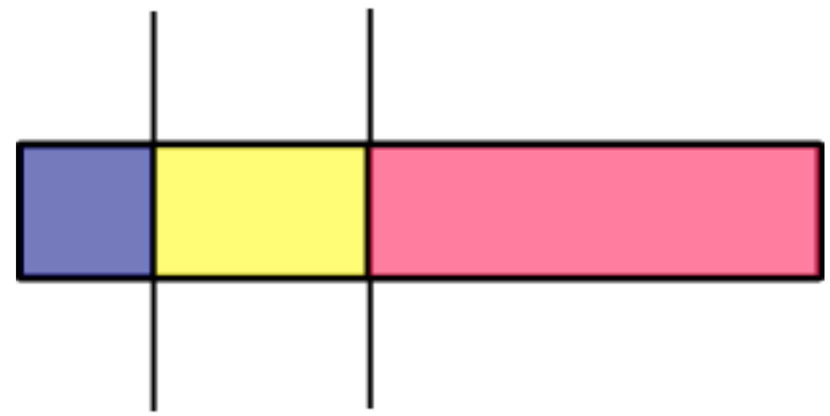
# Existence

- Given cake division $\{A_1, A_2, \ldots, A_M\}$, get an NxM matrix of valuations $[\ \mu_i(A_j)\ ]$
- Dvoretsky-Wald-Wolfowitz(1951): for non-atomic measures $\mu_i$, as divisions vary, the resulting set of matrices is compact & convex.
- Lyapunov: M=2. Neyman: N=M.

# Existence

- Mother Goose:
  "Jack Sprat could eat no fat
  His wife could eat no lean
  So 'twixt them both they cleared the cloth,
  And licked the platter clean."
- Barbanel(1996): if measures independent, can get *super-envy-free:* $\mu_i(A_j) < 1/n$ for $i \neq j$.

# “How?”



- Are there procedures for finding a “fair” division?
- Steinhaus, 1948

# Procedures

- Consider:
  preferences, rules, strategies, outcomes
- Not strategy proof,
  but guarantee fair outcome
  if you follow the procedure

# N=2 people

- "I cut, you choose."
  Fair: Proportional, Envy-free
- Austin's Procedure
  Fair: Equitable
  (and Proportional, Envy-free, too)

# Why is cut-and-choose fair?

- Rules vs [Strategies]:
  - I cut [so I think both pieces half-half]
  - You choose [the one you like best]
- Result is proportional & envy-free

# Austin's Procedure (1982)

- A "moving-knife" procedure
- Player A holds two knives over cake, one at left edge, [so her measure in between is 1/2].
- Move right. Player B calls "cut!" [when measure in between is 1/2].
- Result is equitable, in fact, "perfect"

# N>2
# Proportional Division

- Steinhaus Lone Divider(1948)
- Banach-Knaster Last Diminisher(1948)
- Dubins-Spanier Moving Knife(1961)
- Fink Lone Chooser (1964)
- Even-Paz Divide-and-Conquer (1984)

# Steinhaus Lone Divider(1948)

- Player A cuts cake [so all are 1/3 to her].
- B, C mark piece(s) [good to them].
- If unable to allocate, then let A take unmarked piece, lump other 2 pieces together and let B,C divide that.
- Kuhn(1967) generalized to N>=4.  How?

# Banach-Knaster
# Last Diminisher(1948)

- First player holds knife [so right piece is 1/N].
- Other players take turn: option to diminish right piece [if they feel >1/N].
- Last diminisher gets right piece.

# Dubins-Spanier Moving Knife(1961)

- Referee slowly moves knife rightward.
- Player calls "cut" [when left piece is = 1/N].
- The cutter gets piece to left.

# Fink lone chooser(1964)

- Don't need to know N in advance.
- Player *A* cuts in 2 [equal size], *B* chooses.
- *A*,*B* cut their pieces into 3 [equal], *C* chooses one of each.
- *A*,*B*,*C* cut their pieces into 4 [equal], *D* chooses one of each, etc.

# food for thought

- proportional methods:
  what about entitlements?
  (eg, player *A* deserves 2/3, player *B* 1/3)
- what if cake is undesirable? (chore division)
- how many steps or cuts?
- literature in each area

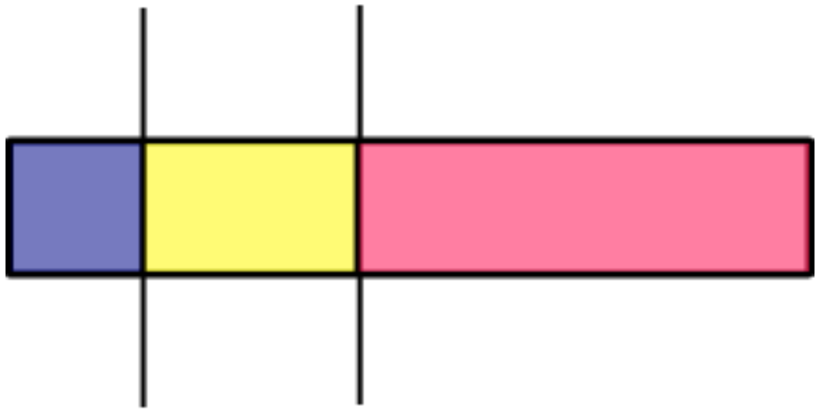
# Even-Paz Divide-and-Conquer(1984)

- O(N *log N*) steps
- Each person marks location of division in *a*:*b* ratio, where *a*=floor(*n*/2), *b*=ceiling(*n*/2).
- Cut at *a*-th mark, leftmost *a* people divide left piece, others divide right, inductively.
- Edmonds-Pruhs(2011): any finite proportional *protocol* takes $\Omega(N \log N)$

# Envy-free methods

- 3-Person: Selfridge-Conway (1960)
- 3-Person: Stromquist moving-knife (1980)

# Selfridge-Conway method



- A cuts [into thirds]
- B trims [to make 2-way tie for largest], sets aside trimmings
- C chooses, then B (who is required to take trimmed piece if C didn't), then A gets last piece.
- What to do with trimmings?

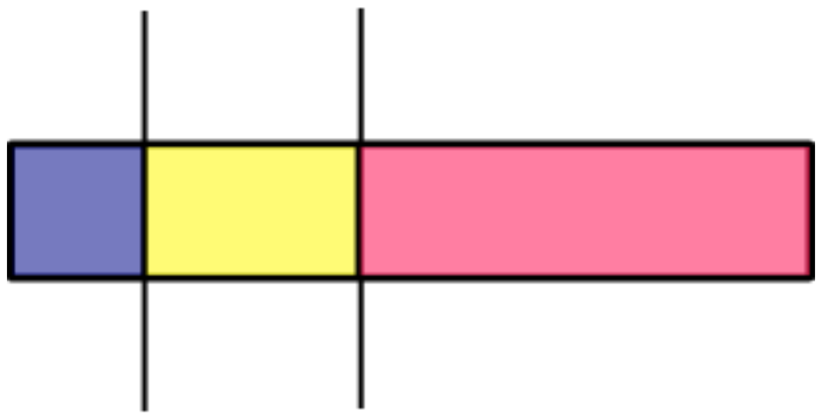
# Selfridge-Conway(1960)

- For trimmings:
- Call T the one who got trimmed piece, and N the other.
- Let N divide [into thirds]. Let T choose, then A, then N gets last piece.

# Stromquist Moving Knife(1980)

- Referee: holds one knife at left edge, moves right
- 3 players hold knives such that…
- When player calls ‘cut’: cake is cut at middle knife, left piece goes to caller…
  [Player calls when leftmost piece is largest.]

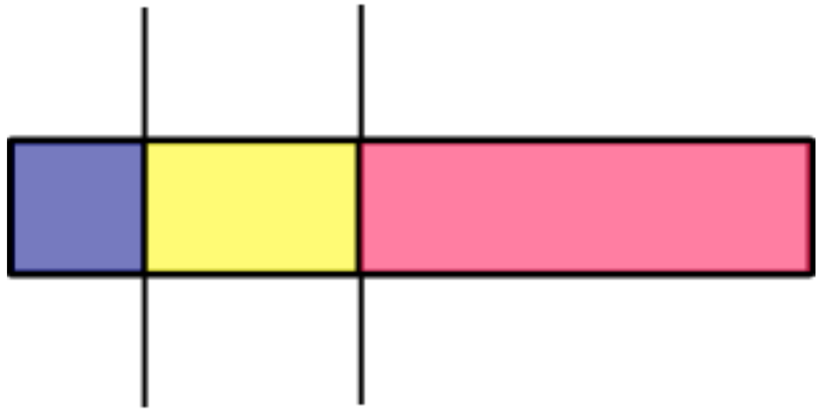
# Brams-Taylor (1995)



- First N-person envy-free procedure
- Finite, but unbounded
- Decimates the cake too

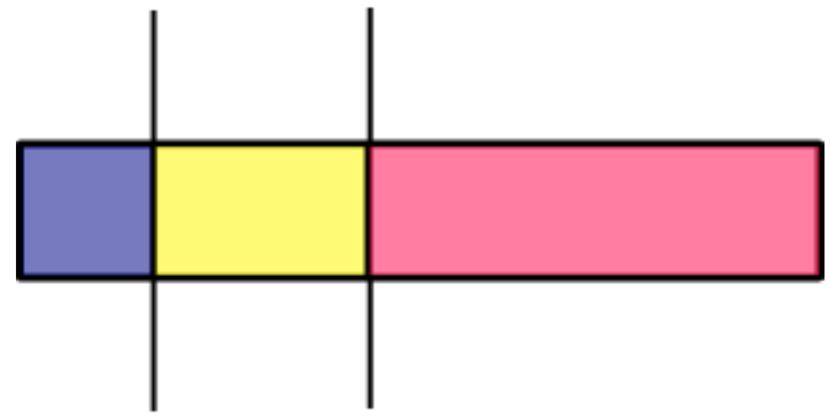
- Aziz-MacKenzie(2017): bounded procedure!

# Contiguous Pieces?

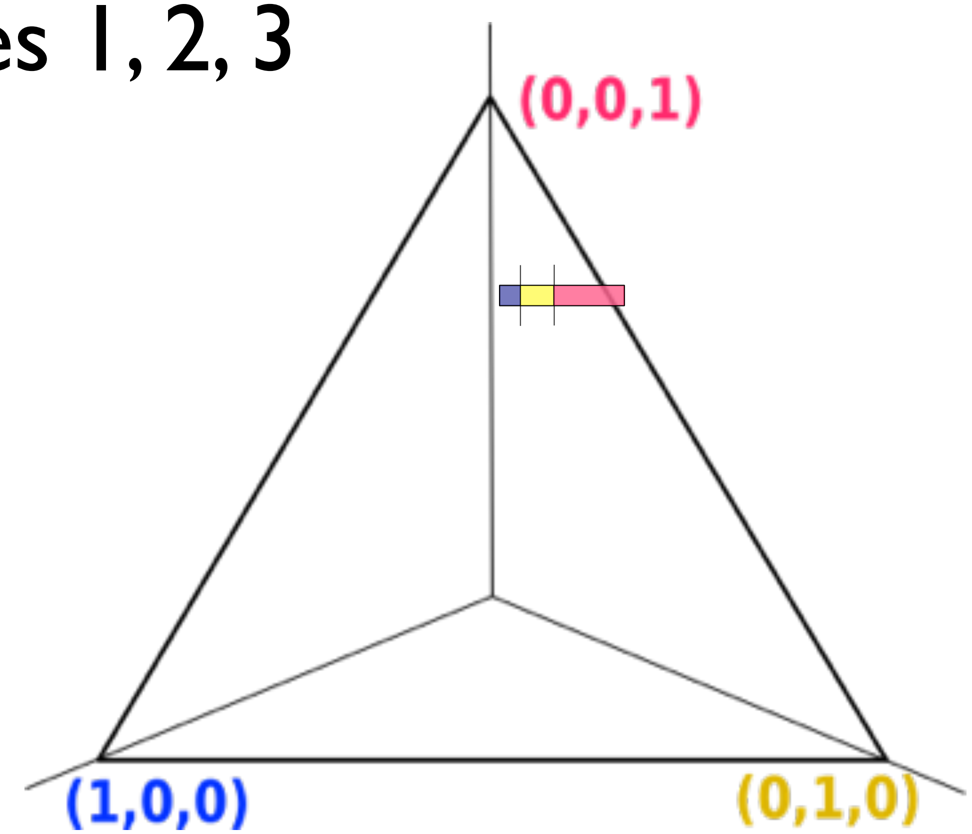


- Is there an envy-free, finite protocol for 3 or more players? Stromquist(2008): No.
- Stomquist(1980) achieves contiguous pieces, but not a protocol.
- Yes, if allow *approximate* envy-free.
  Converge to solution.

# The Space of Cake Divisions



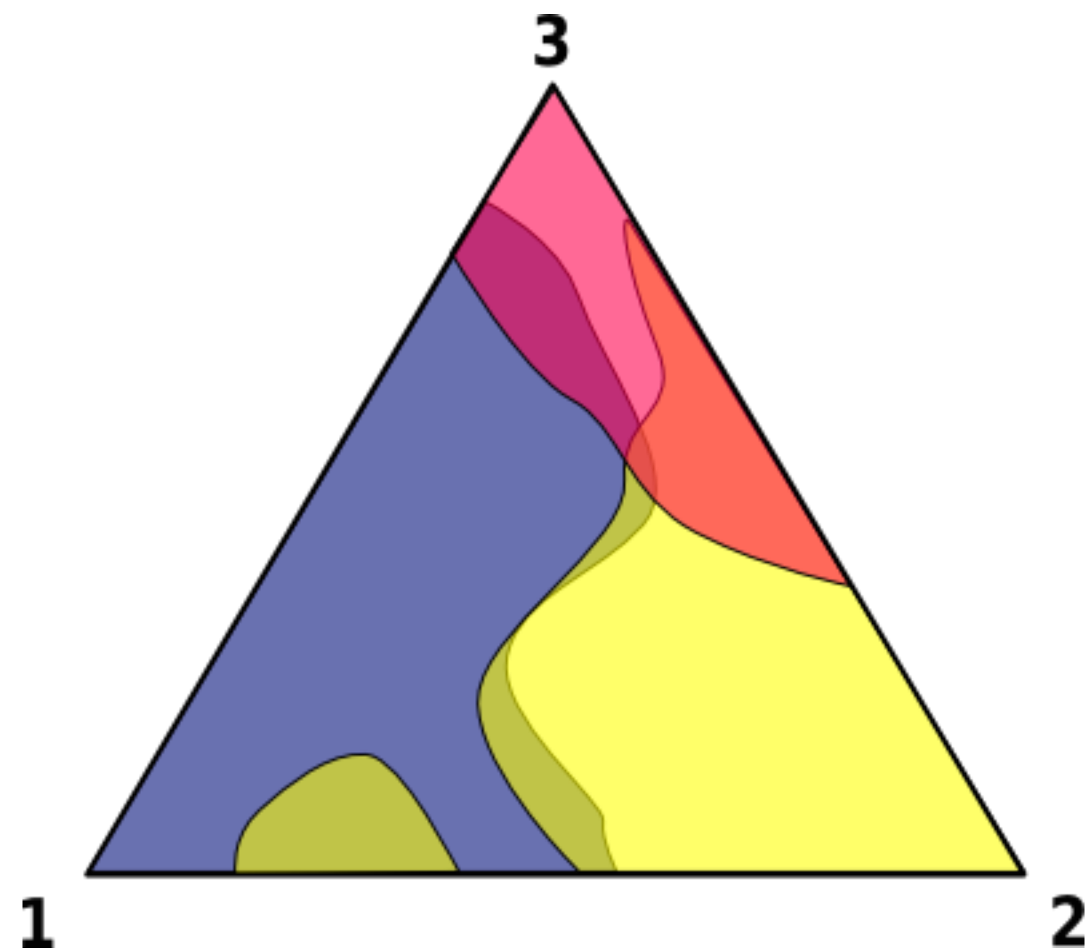
- Each division of cake is a triple of numbers (x, y, z) representing widths of pieces 1, 2, 3
- A point in a triangle!



# Preferences

- Sets where Piece 1, 2, 3 preferred

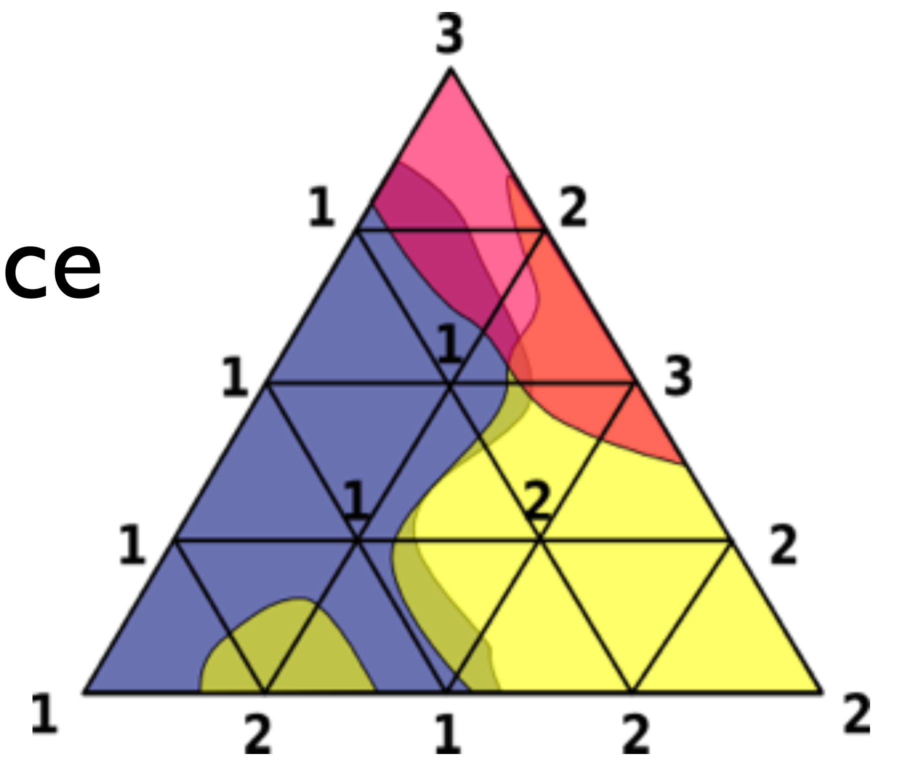
# KKM Lemma




- Sets 1,2,3 intersect
- KKM-Gale: some permutation overlaps

# Combinatorial Topology

- View the space of divisions as a geometric/topological space
- Triangulate the space
- Label vertices by preferences
- Use labels to find a 'good' solution



# Sperner's Lemma

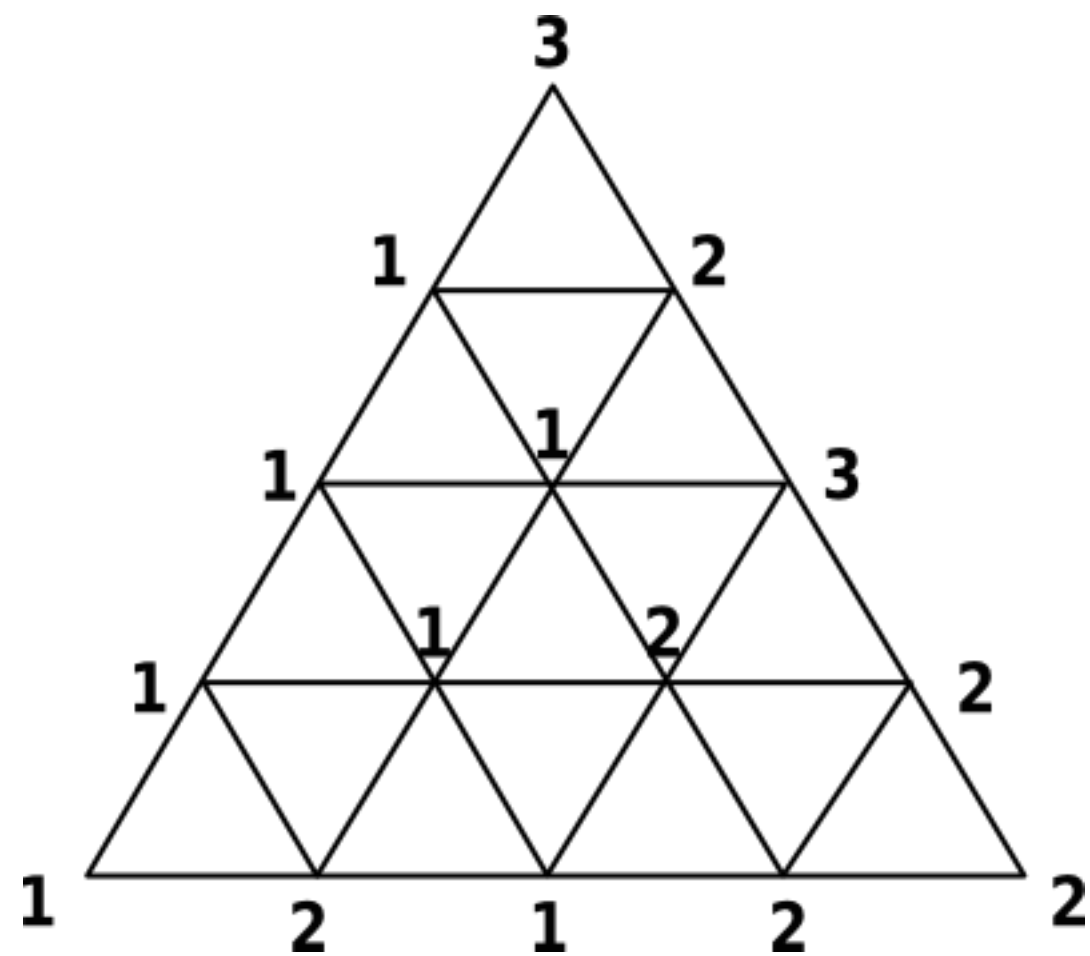
- applications: fixed point theorems, solutions to differential equations
- Brouwer fixed point theorem

# Sperner's Lemma

- Draw a triangle & break it into lots of baby triangles.

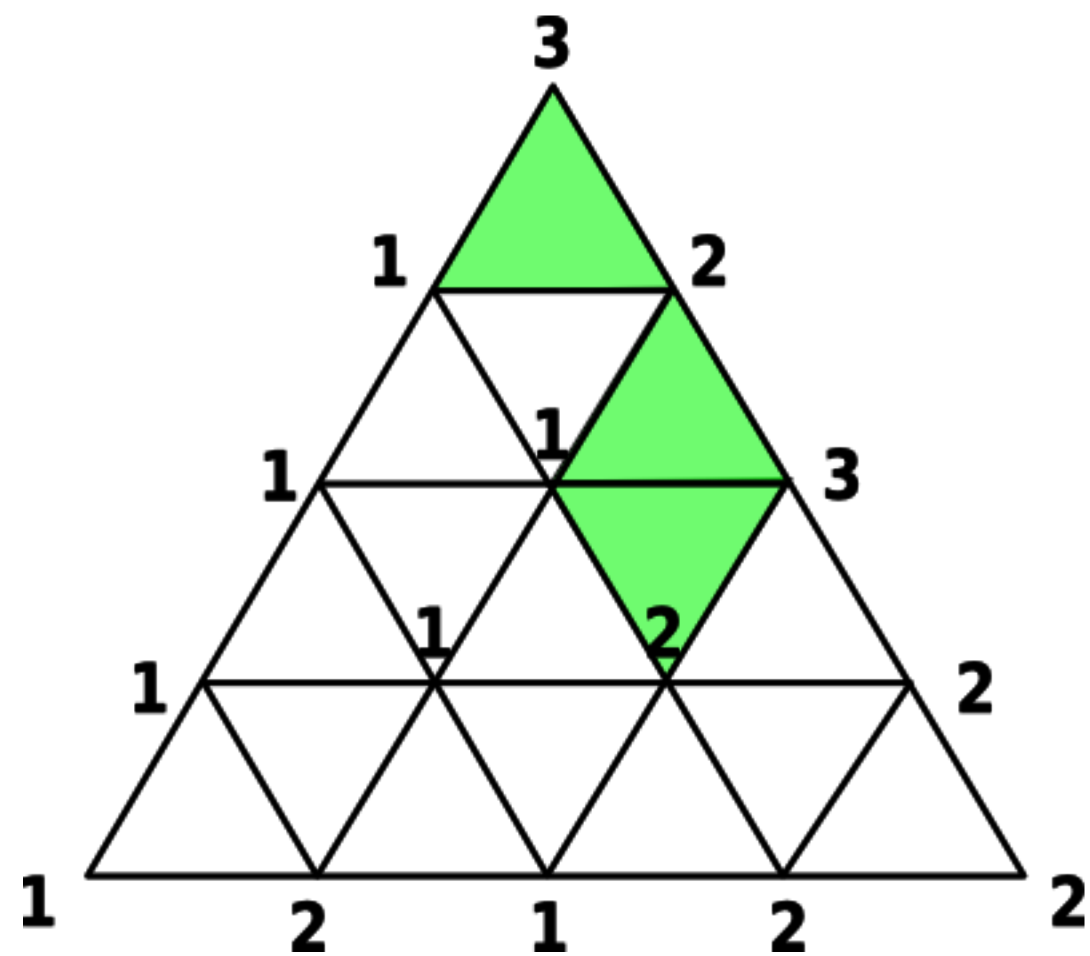
# Sperner's Lemma

- A Sperner-labelled triangulation of a triangle must ...



# Sperner's Lemma

- A Sperner-labelled triangulation of a triangle must have an odd number of baby 123-triangles.



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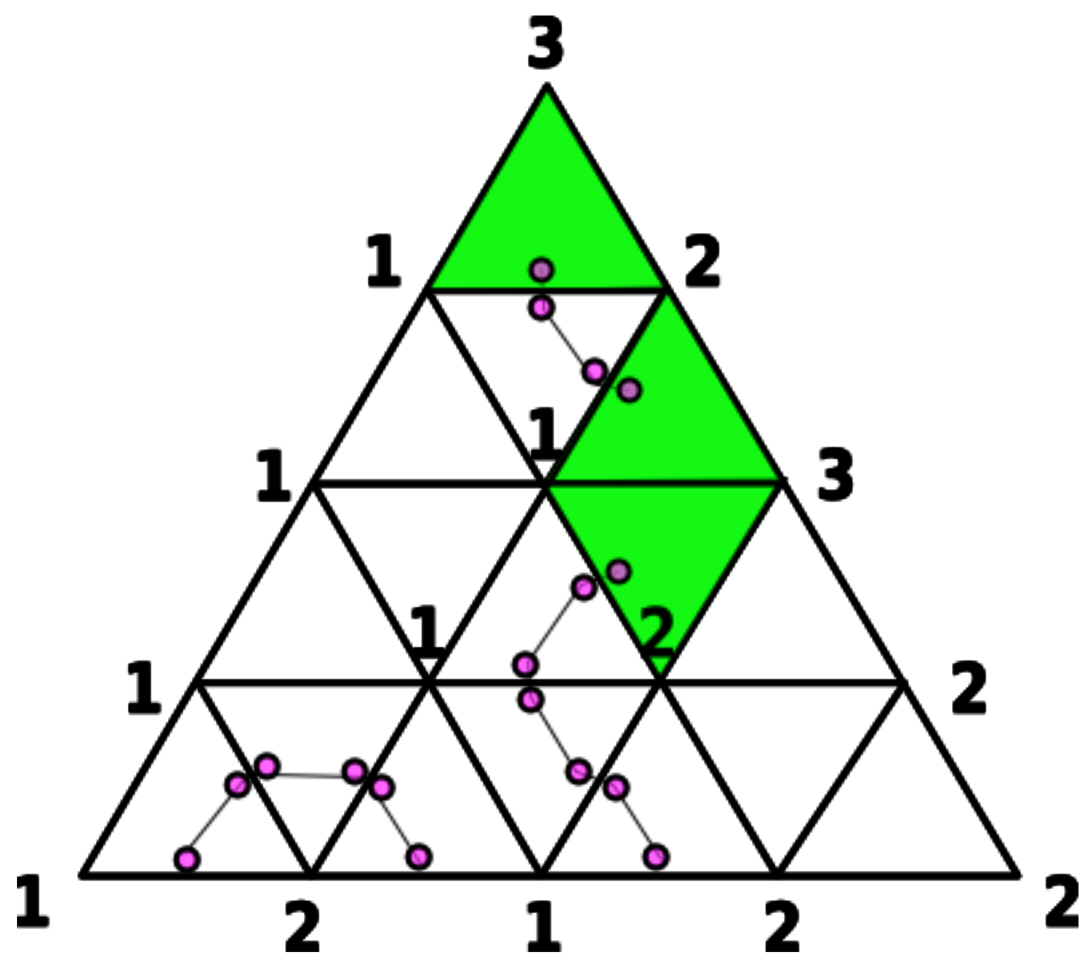
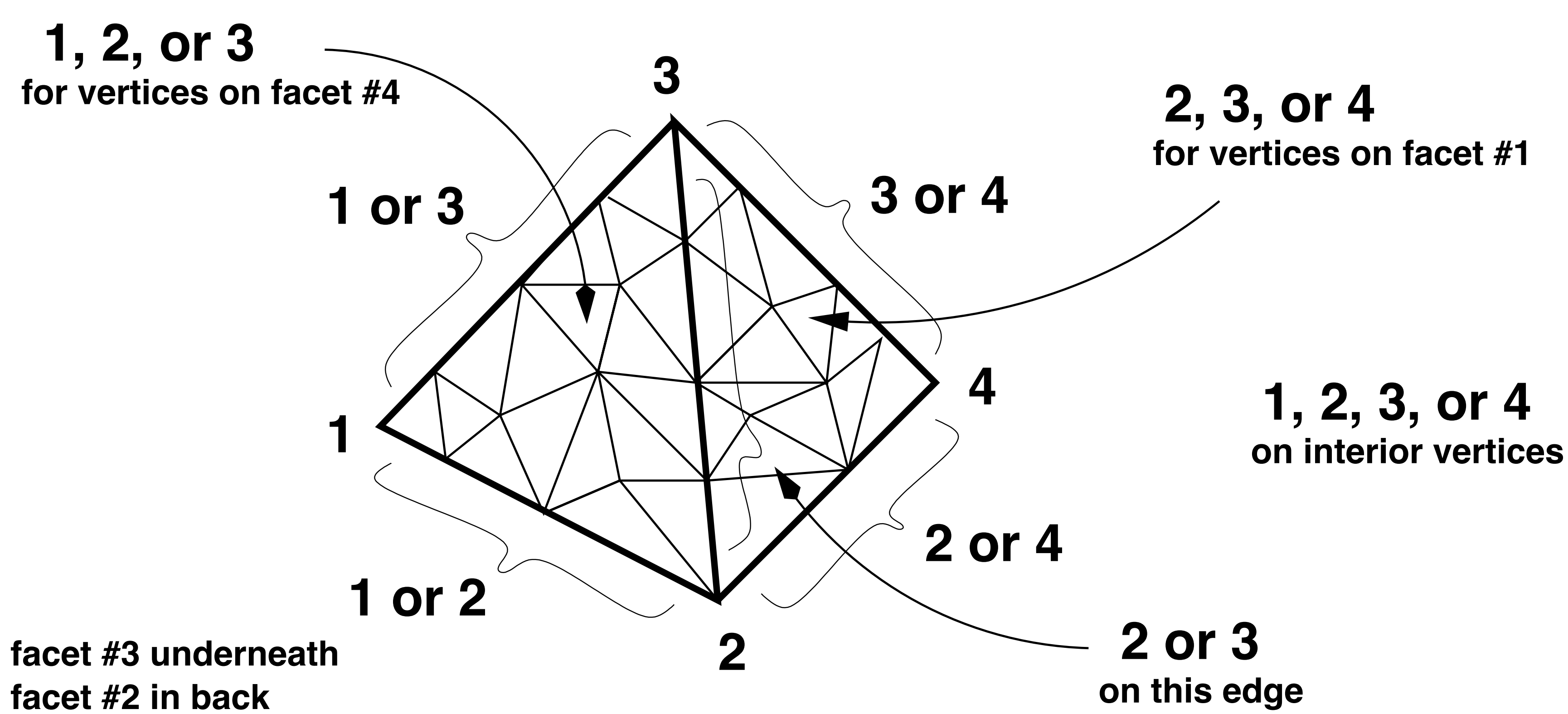


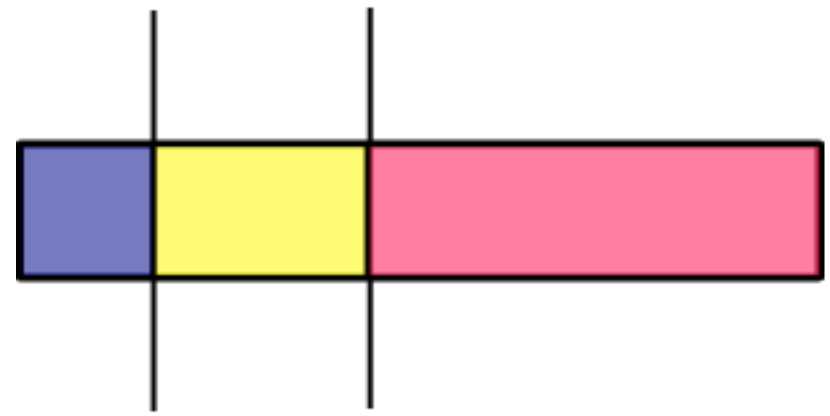
# Sperner's Lemma

- More dimensions: proofs are similar
- Follow 123-doors to baby (1234)-tetrahedra



# Cake

# The Space of Cake Divisions



- Each division of cake is a triple of numbers (x, y, z) representing widths of pieces 1, 2, 3
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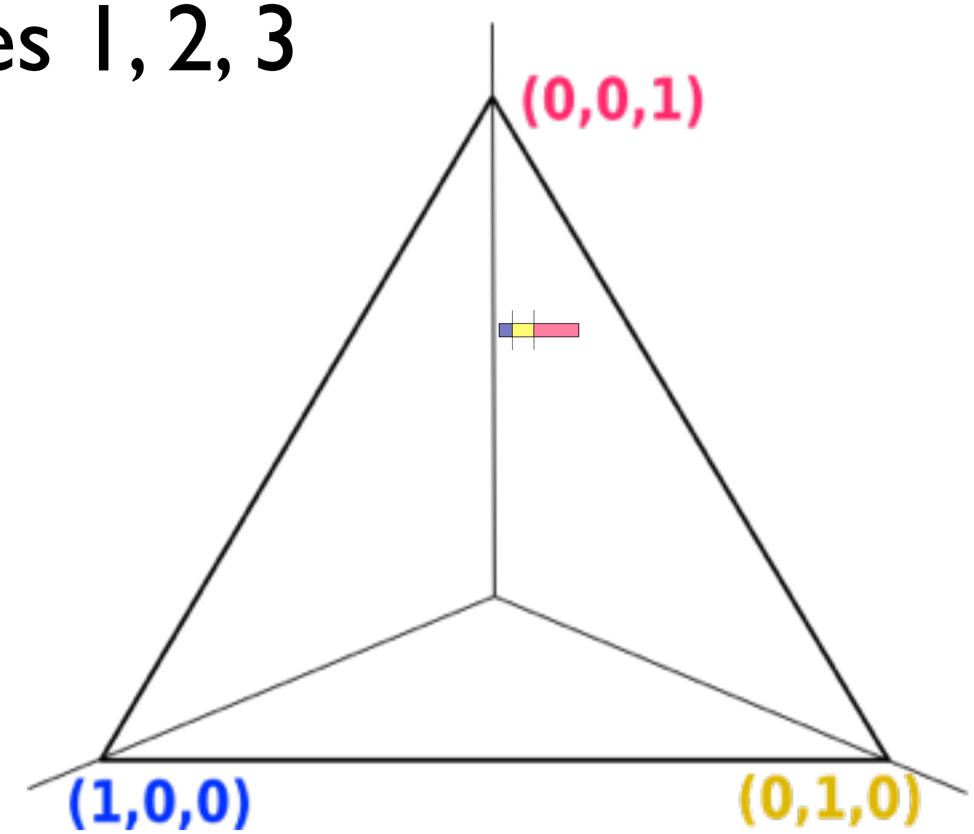
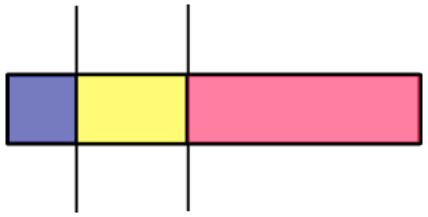
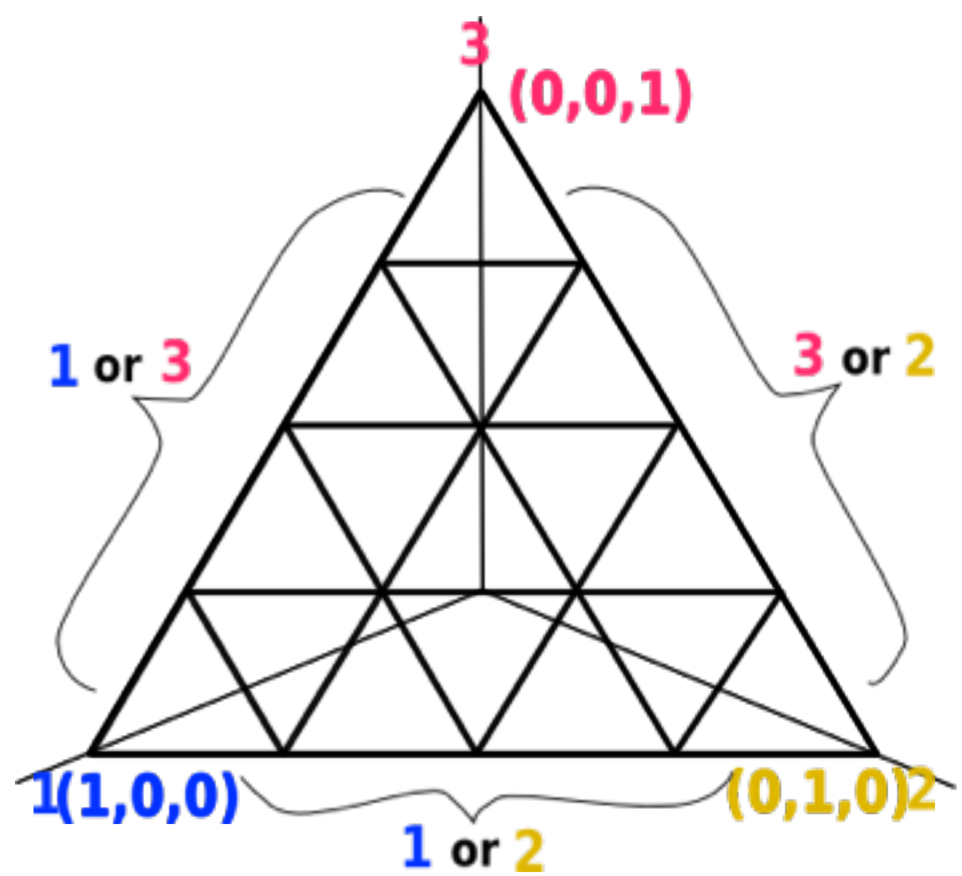




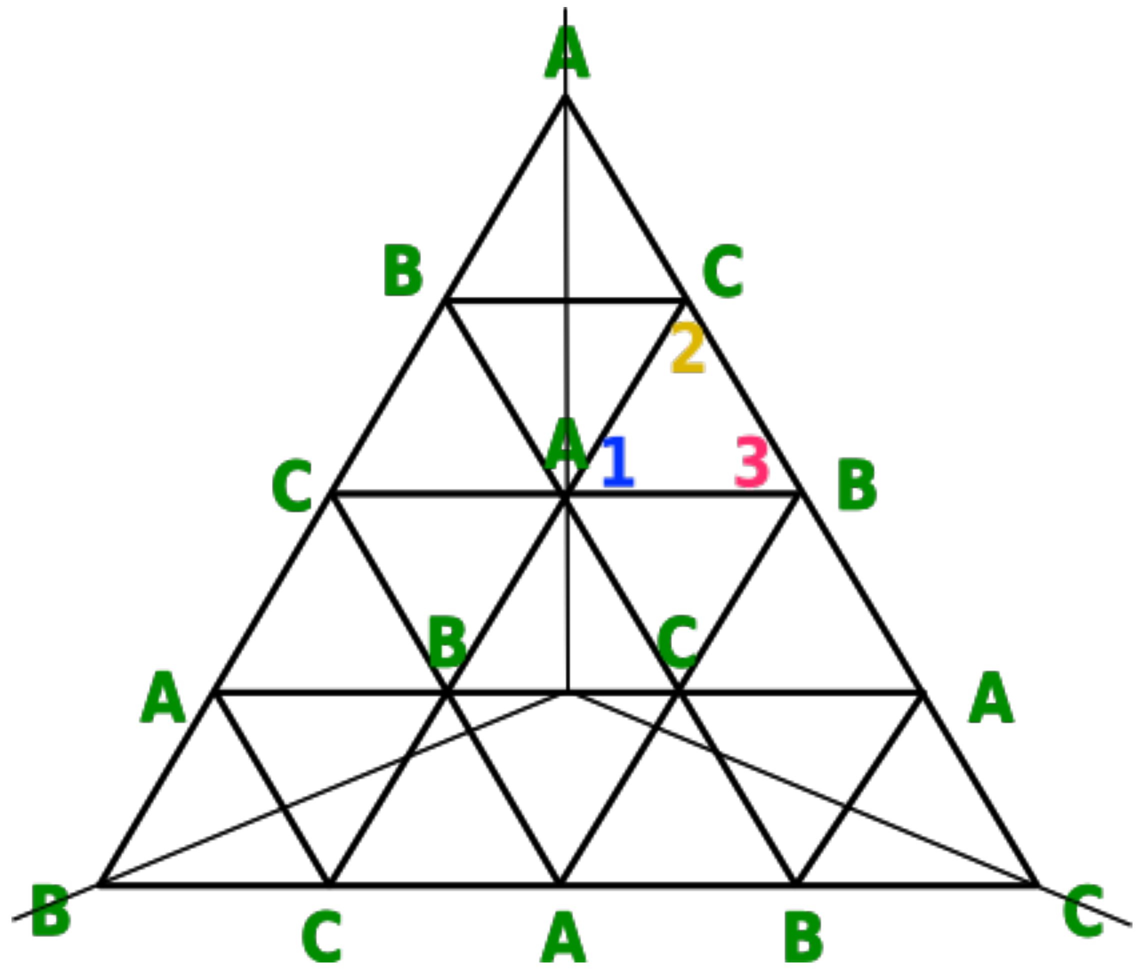
# Triangulate!

- Each vertex is a division. Assign "owners".
- So ask owner at each vertex: "which piece would you choose in this division?"
- Answers are Sperner labels!

Get 123-triangle (somewhere)



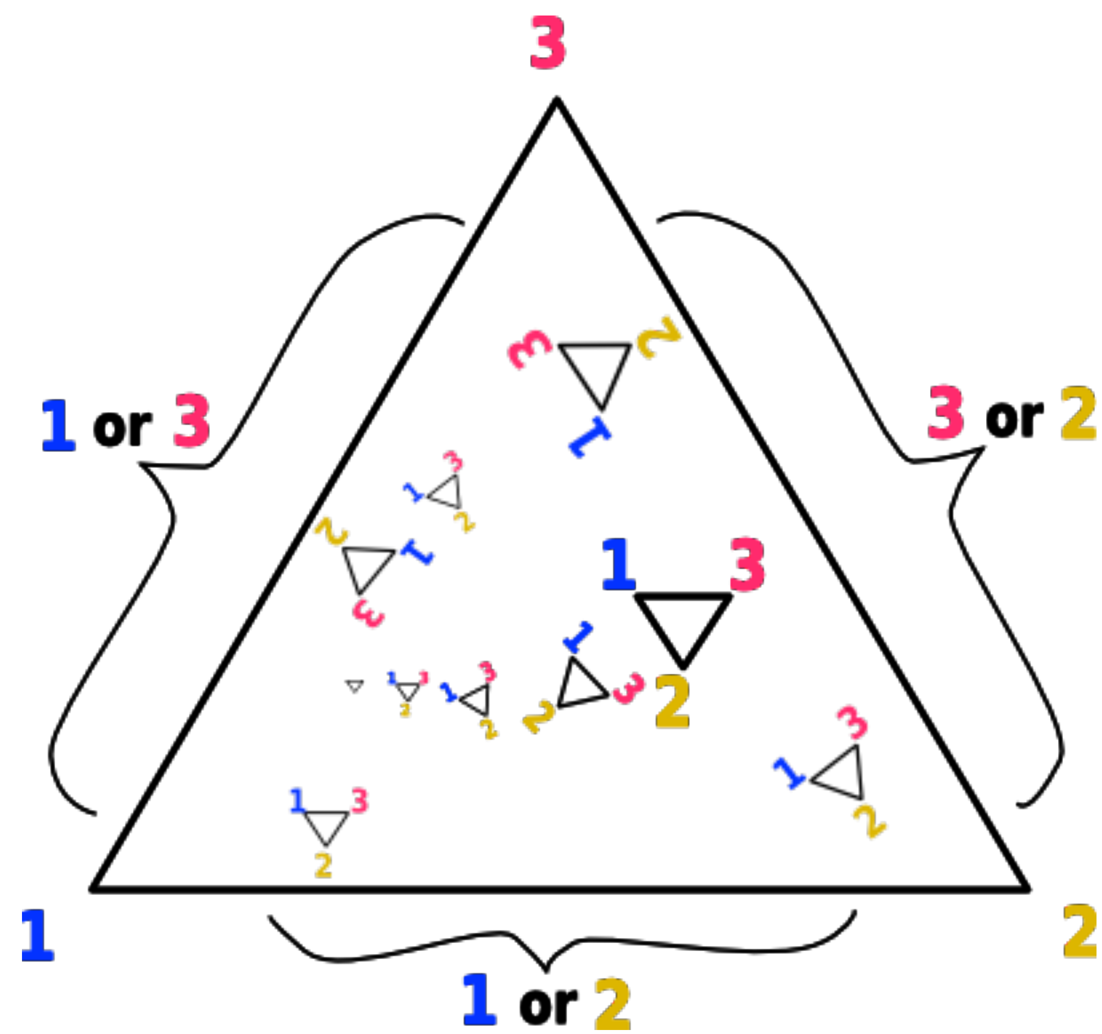
# But if the owners were assigned like this...




- Then any 123-triangle came from A,B,C responses
- Any pt inside is an approx envy-free division!

# Towards exact envy-free

- Repeat for smaller triangulations
- Get sequence of finer triangles
- They converge to some point!




That point is an envy-free division!

# Cake Cutting Thm

If these conditions hold:

- (Hungry Players) - no one would take an empty piece
- (Closed Preferences) - preferences unchanged by limits

there exists an envy-free division.

- Simmons (1980): constructive approx envy-free method
- Deng-Qi-Saberi(2012): analyze complexity

# Dividing a Comic Book

- *The Simpsons: Three Men and a Comic Book*, 5/9/1991

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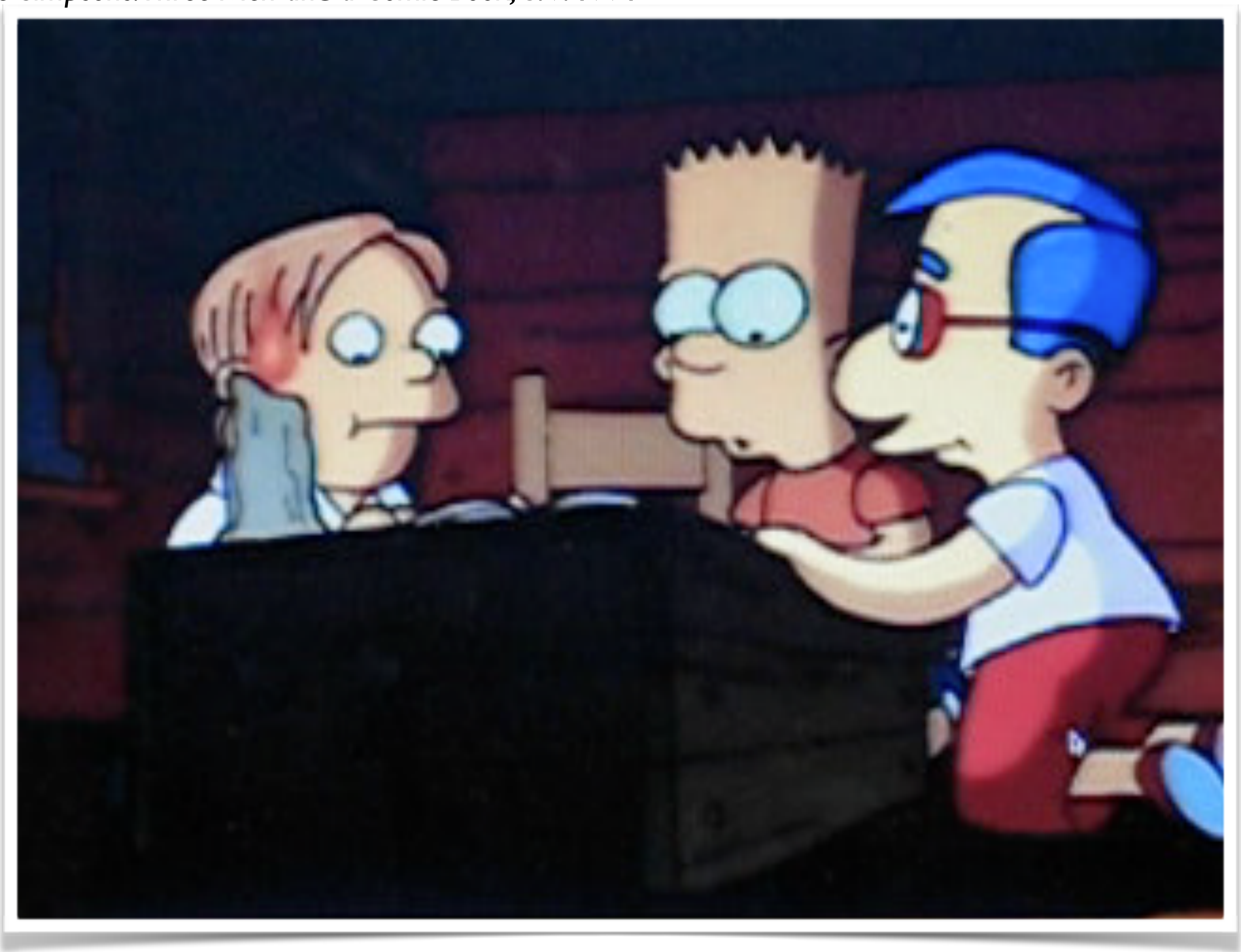
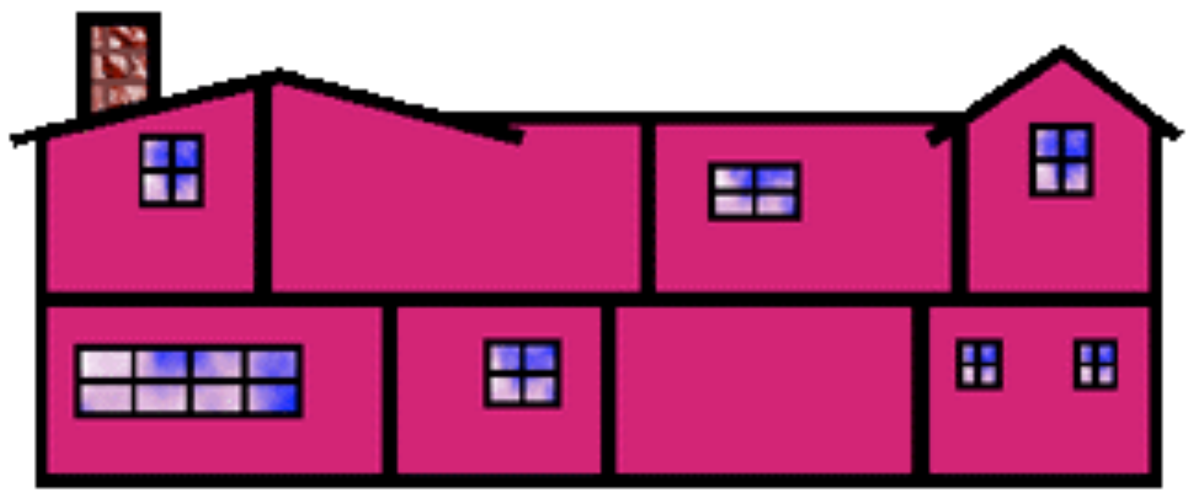
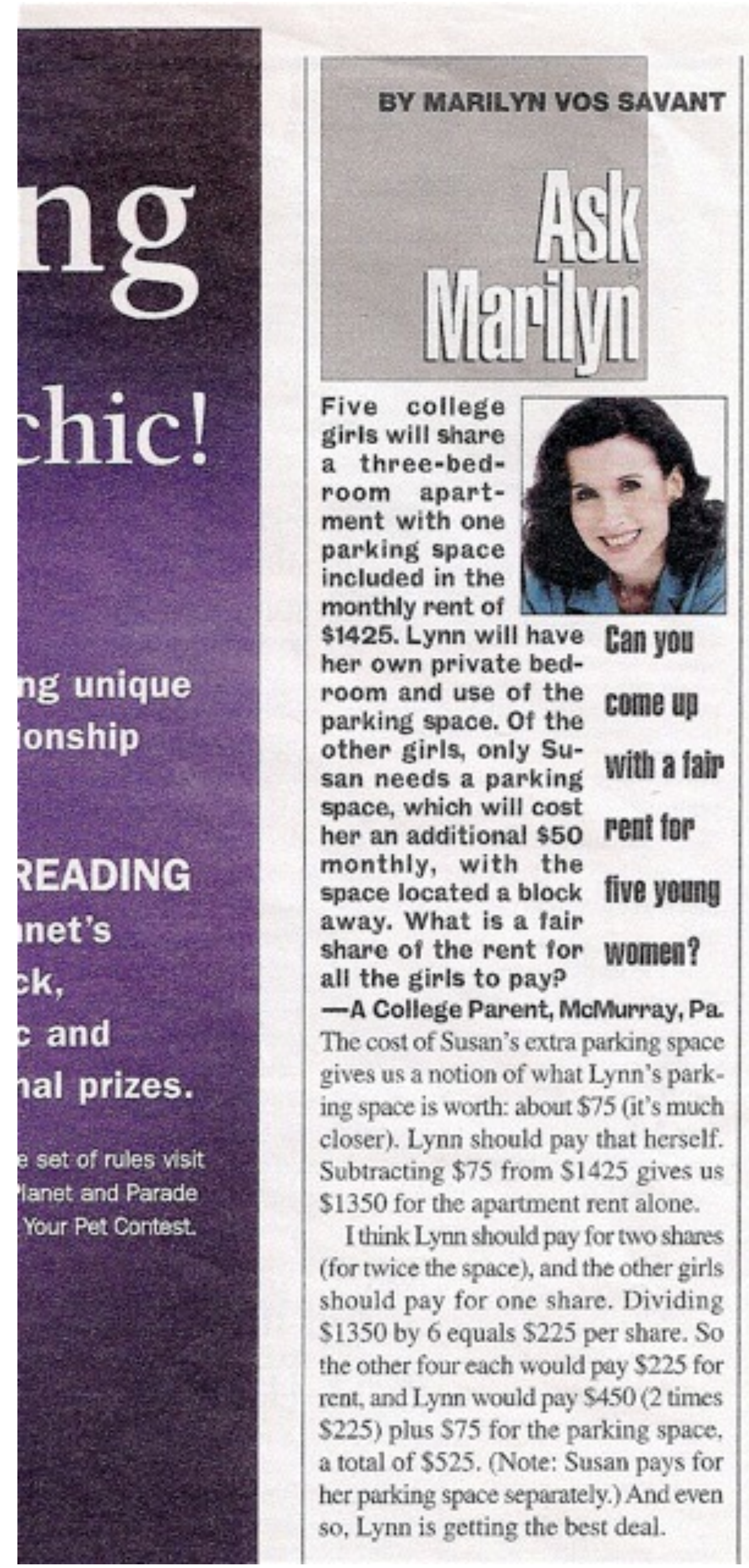


# What about rent division?



# Ask Marilyn?

- 7/14/2002, Parade Magazine





BY MARILYN VOS SAVANT

**Ask Marilyn**

Five college girls will share a three-bedroom apartment with one parking space included in the monthly rent of $1425. Lynn will have her own private bedroom and use of the parking space. Of the other girls, only Susan needs a parking space, which will cost her an additional $50 monthly, with the space located a block away. What is a fair share of the rent for all the girls to pay?

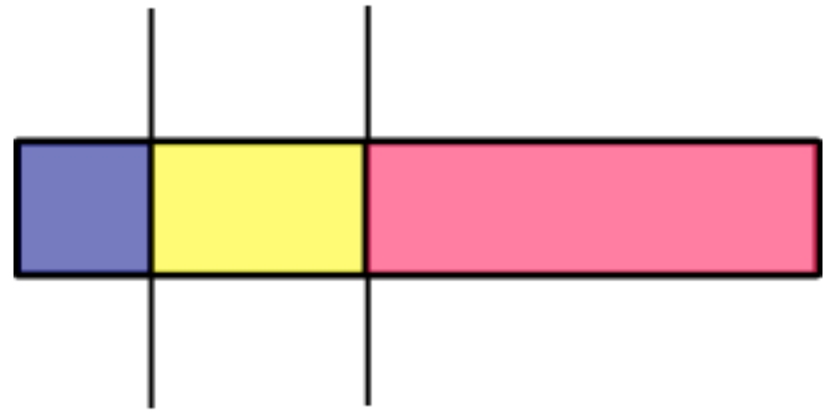
—A College Parent, McMurray, Pa.

**Can you come up with a fair rent for five young women?**

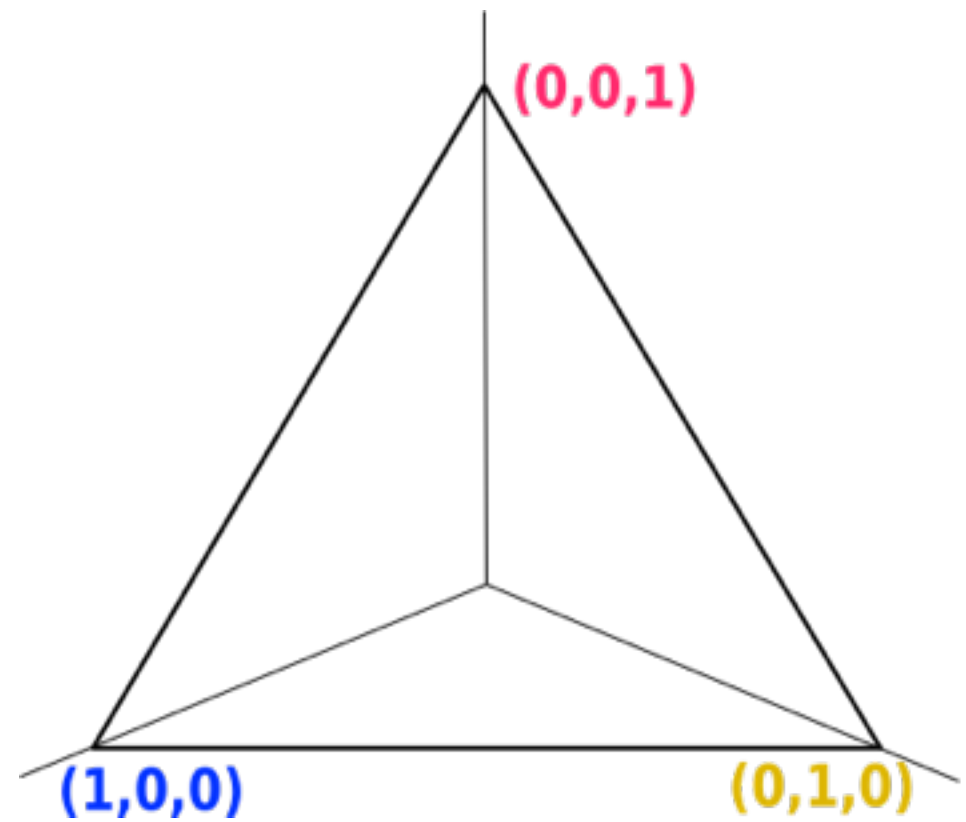
The cost of Susan's extra parking space gives us a notion of what Lynn's parking space is worth: about $75 (it's much closer). Lynn should pay that herself. Subtracting $75 from $1425 gives us $1350 for the apartment rent alone.

I think Lynn should pay for two shares (for twice the space), and the other girls should pay for one share. Dividing $1350 by 6 equals $225 per share. So the other four each would pay $225 for rent, and Lynn would pay $450 (2 times $225) plus $75 for the parking space, a total of $525. (Note: Susan pays for her parking space separately.) And even so, Lynn is getting the best deal.

# The Space of Rent Divisions



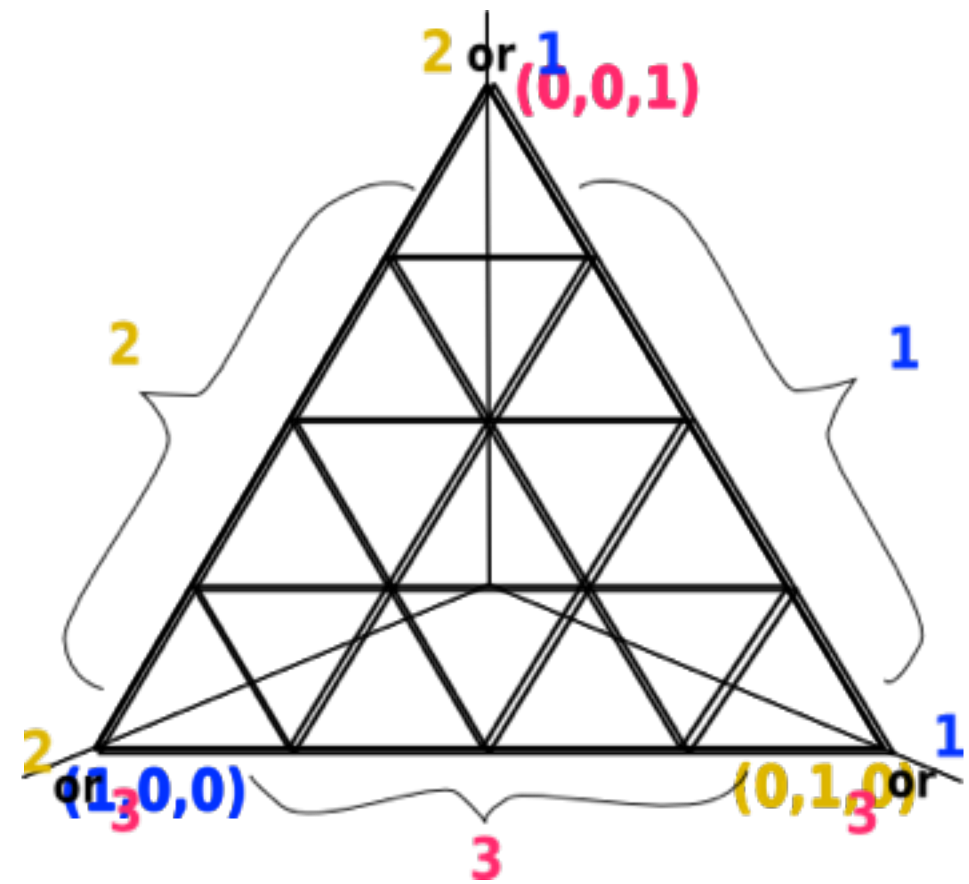
- Each division of rent is a triple of numbers (x, y, z) summing to total rent.
- A point in a triangle!



# Rent Division

- Each vertex is a division. Assign "owners".
- So ask owner at each vertex: "which room would you prefer?"

Dual! Get 123-triangle!

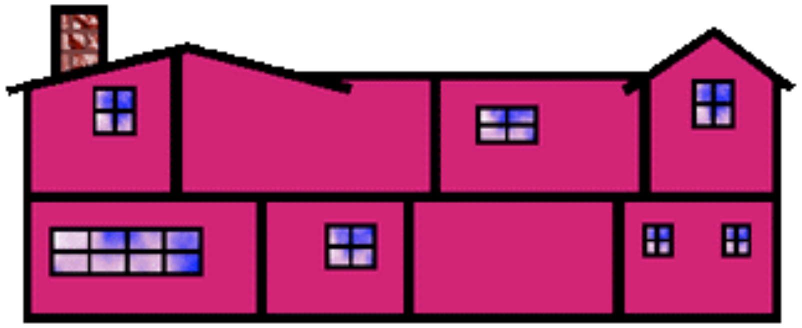


# Rental Harmony Thm (S., 1999)

If these conditions hold:

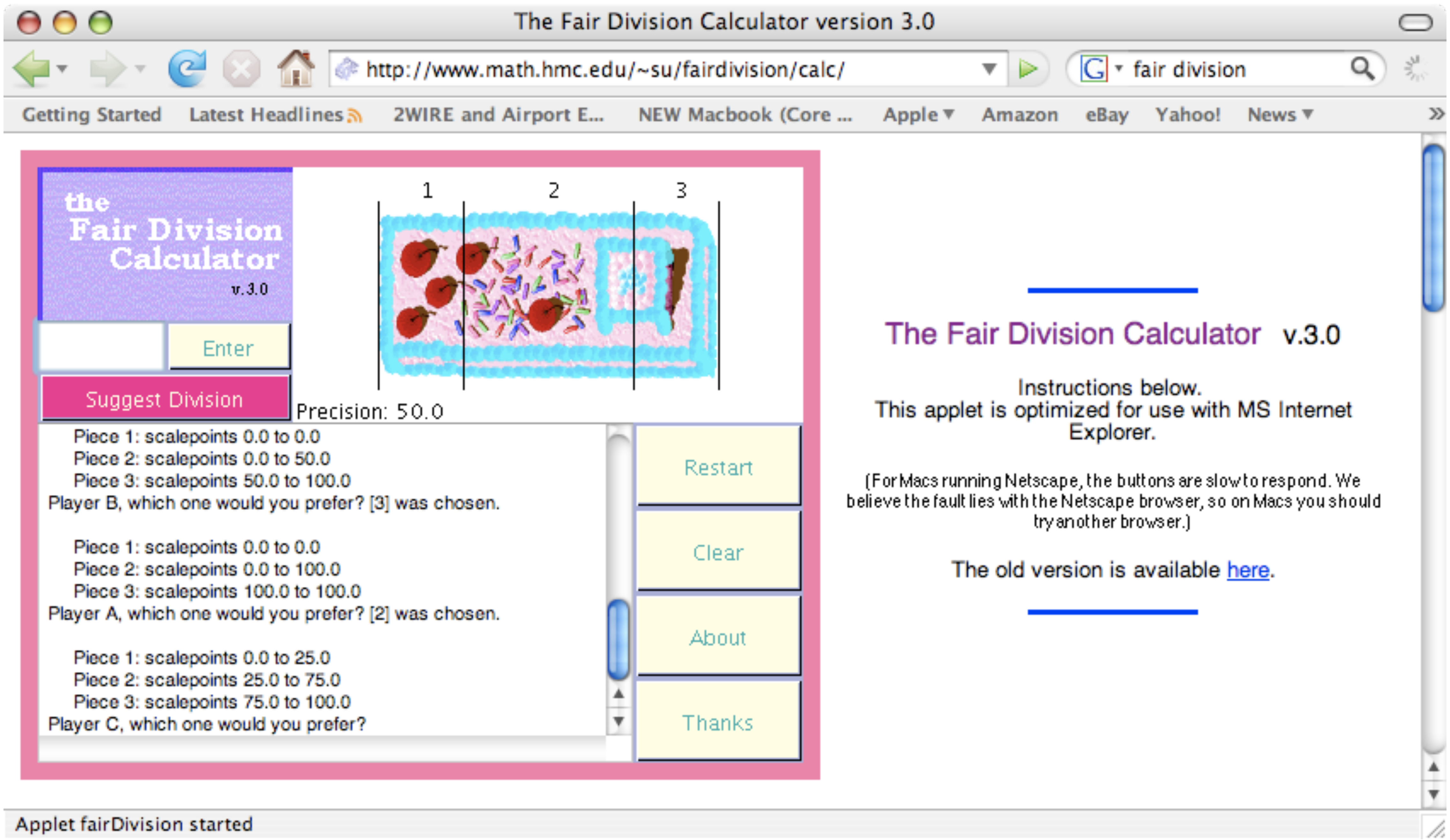
- (Good House) - in every division, some room will be chosen
- (Miserly Tenants) - no one would pass up a free room
- (Closed Preferences) - room preference unchanged by limits

then there exists an envy-free rent division.



- Azriela-Shmaya(2014): rooms shared
- Frick-HoustonEdwards-Mueunier(2017): a secret pref

# The Fair Division Calculator



4/28/2014

# New York Times version



The New York Times

SCIENCE

## *To Divide the Rent, Start With a Triangle*

By ALBERT SUN APRIL 28, 2014




Last year, two friends and I moved into a small three-bedroom apartment in Manhattan. We chose it for its relatively reasonable price — around $3,000 a month — and its convenient location. Just finding it was a challenge, but then we faced another one: deciding who would get each bedroom.

The bedrooms were different sizes, ranging from small to very small. Two faced north toward the street and had light; the third and smallest faced an alley. The largest had two windows; the midsize room opened onto the fire escape.

Every month, unrelated people move into apartments together to save on rent. Many decide to simply divide the rent evenly, or to base it on bedrooms' square footage or perhaps even on each resident's income.

But as it turns out, a field of academics is dedicated to studying the subject of fair division, or how to divide good and bad things fairly among groups of people. To the researchers, none of the typical methods are satisfactory. They have better ways.

The problem is that individuals evaluate a room differently. I care a lot about natural light, but not everyone does. Is it worth not having a closet? Or one might care more about the shape of the room, or its proximity to the bathroom.



A division of rent based on square feet or any fixed list of elements can't take every individual preference into account. And negotiation without a method may lead to conflict and resentment.

I set out to find a better way to divide our rent. That's how I came across a paper by Francis Su, a math

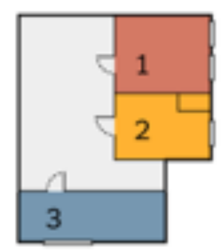
### Sperner's Lemma and Rental Harmony

A mathematical theorem called Sperner's Lemma can be used to divide unequal assets fairly.

**The Problem**

Three friends **Ashwin**, **Bret** and **Chad** want to share an apartment.

The total rent is $3,000 but the rooms are different sizes. How can they choose rooms and divide the rent fairly?



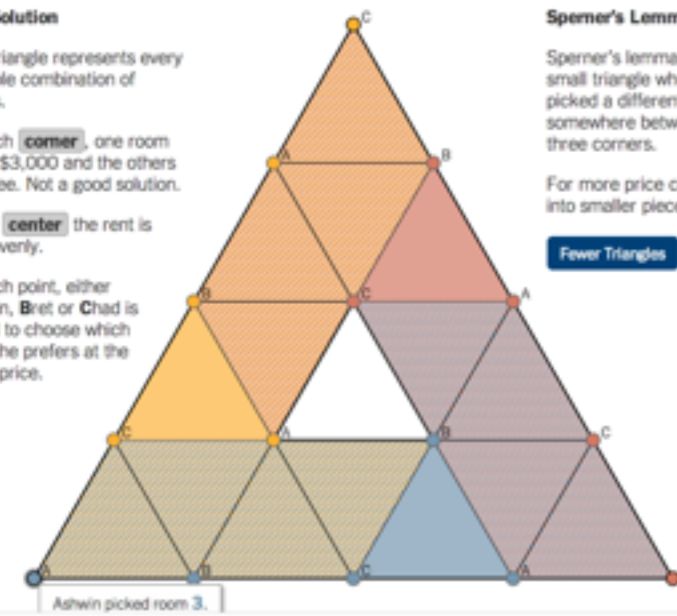

**The Solution**

This triangle represents every possible combination of prices.

At each **corner**, one room costs $3,000 and the others are free. Not a good solution.

In the **center** the rent is split evenly.

At each point, either **A**shwin, **B**ret or **C**had is asked to choose which room he prefers at the given price.



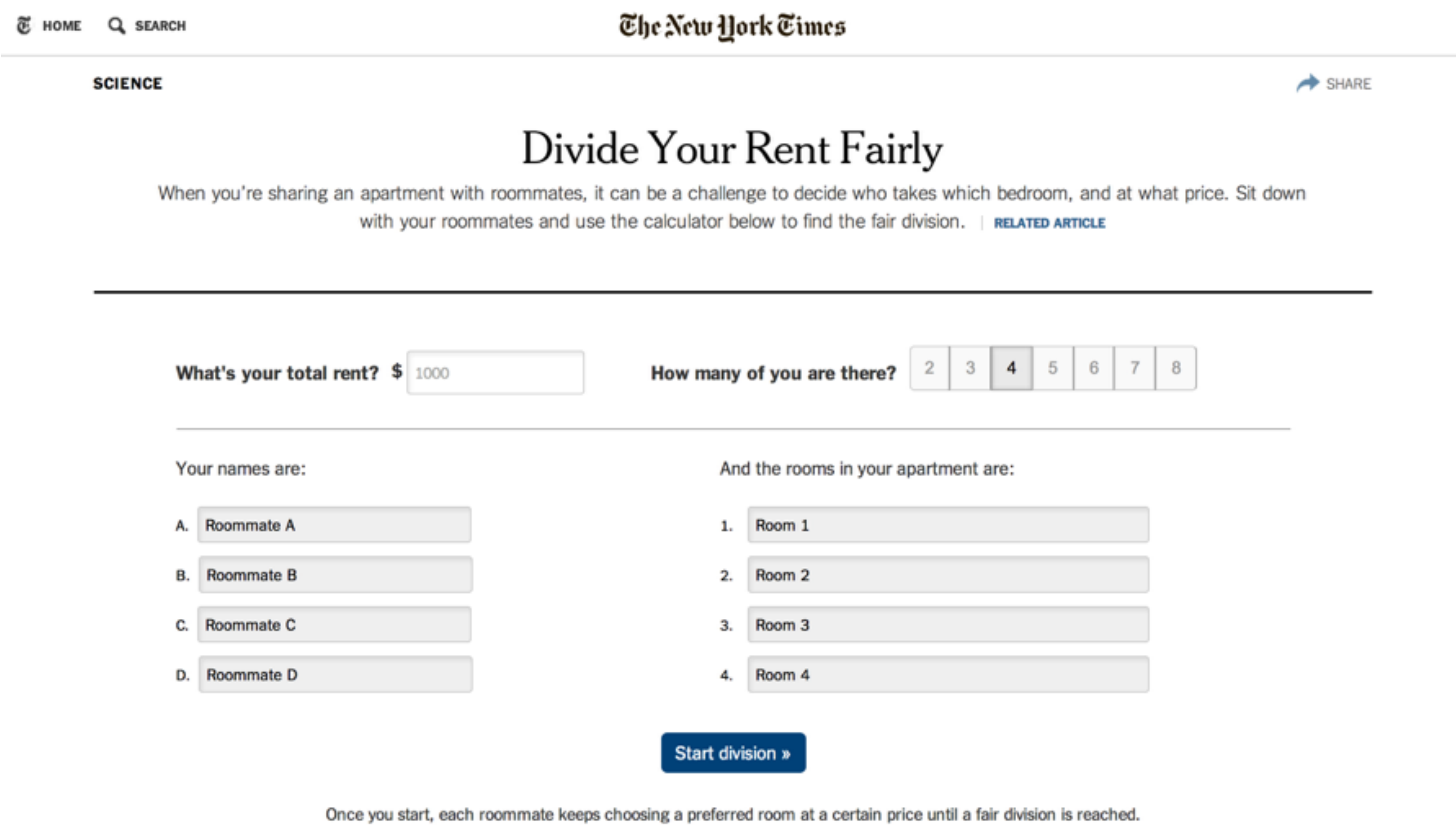

**Sperner's Lemma**

Sperner's lemma guarantees that there is a small triangle where every roommate has picked a different room. The "fair" price lies somewhere between the prices at those three corners.

For more price choices, divide the triangle into smaller pieces:

Fewer Triangles More Triangles

# New York Times version



- From a user: “Thank you kindly for your work and for publishing working calculator online. It has certainly made our lives better.”

- Vary the meanings of these words:
  “cake” “cut” “fair” “how”
- Ask different questions:
  existence? construction? properties?

# Starting Points

- Brams-Taylor: *Fair Division: from Cake Cutting to Dispute Resolution*
- Robertson-Webb: *Cake-Cutting Algorithms*
- Barbanel: *The Geometry of Efficient Fair Division*