Fair Division of Manna: a brief survey

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full responsibility for(private) individual ordinal preferences: tastes", not needs

resources are common property (inheritance, bankruptcy, divorce), agents have identical "rights" , no individual responsibility for the creation of the resources

pure consumption of desirable goods/ commodities consumed privately

general ArrowDebreu preference domain; subdomains of preferences: homothetic, linear, Cobb Douglas, Leontief and dichotomous preferences

variants: non disposable single commodity; assignment with lotteries

(see [17] or [25] for a survey)

goal: to design a **single-valued, easily computed** division rule achieving

Efficiency (aka Pareto optimality) (EFF)

Fairness: difficult to define, sanctioned by a handful of tests

and if possible

Strategyproofness (SP): truthful report of one's preferences (dominant strategy for a prior-free context)

or a weaker implementation property

two versions of horizontal equity

Equal Treatment of Equals (ETE): same utility for same preferences

strengthened as

Anonymity (ANO): symmetric treatment of all players (names do not matter) *four tests of fairness*

two single profile tests

Unanimity Lower Bound (ULB): my utility should never be less than the utility I would enjoy if every preferences was like my own (and we were treated equally)

$\longrightarrow$ in the standard Arrow-Debreu model below, the unanimity utility level corresponds precisely to the consumption of 1/n-th of the resources

No Envy (NE): I cannot strictly prefer the share of another agent to my own share

ETE, ULB, and NE are not logically independent:

$\longrightarrow$ NE implies ETE

$\longrightarrow$ NE implies ULB in the Arrow Debreu model with only two agents, or with linear preferences and any number of agents

there is also a link between $\{$ETE + SP$\}$ and NE two *multi-profile* tests

Resource Monotonicity (RM): when the manna increases, ceteris paribus, the utility of every agent increases weakly

Population Monotonicity (PM): when a new agent is added to the participants , ceteris paribus, the utility of every agent decreases weakly

an incentive (as opposed to a normative) interpretation:

$\longrightarrow$ absent RM, I may omit to discover new resources that would benefit the community

$\longrightarrow$ absent PM, I may omit to reveal that one of us has no right to share the resources

RM and PM insist on *Solidarity* in reaction to shocks on resources or population

ULB and NE define limited yet precise individual rights *a powerful rationality axiom akin to the separability of a social welfare ordering*

Consistency (CSY): when an agent leaves, and takes away the share assigned to him, the rule assigns the same shares in the residual problem (with one less agent and fewer resources) as in the original problem

unlike the other six axioms CSY conveys no intuitive account of fairness; it simply checks that "every part of a fair division is fair"

$\longrightarrow$ each multiprofile test by itself is compatible with grossly unfair rules

the *fixed priority rules* is RM, PM, and CSY: fix a priority ordering of all the potential agents, and for each problem involving the agents in N, give all the resources to the agent in N with highest priority

this is not true for ETE, ANO, ULB, or NE: each property by itself, guarantees some level of fairness

Arrow Debreu (AD) preferences

the canonical microeconomic assumptions

$N \ni i$: agents, $|N| = n$

$A \ni a$: goods, $|A| = K$

$\omega \in \mathbb{R}^K_+$: resources to divide (infinitely divisible)

$\succsim_i$: agent i's preferences: monotone, convex, continuous, hence representable by a continuous utility function

$\rightarrow$ an allocation $(z^i, i \in N)$ is feasible if $z^i \in \mathbb{R}^K_+$ and $\sum_N z^i = \omega$

$\rightarrow$ it is efficient iff the upper contour sets of $\succsim_i$ at z^i are supported by a common hyperplane the equal division rule ($z^i = \frac{1}{n}\omega$ for all i) meets **all axioms** above, except EFF

first impossibility results: fairness $\leftrightarrow$ *efficiency tradeoff*

- EFF $\cap$ ULB $\cap$ RM =EFF $\cap$ NE$\cap$ RM =$\varnothing$ ([18])

the easy proof rests on preferences with strong complementarities, like Leontief preferences

second impossibilities: tradeoff efficiency $\leftrightarrow$ *strategyproofness* $\leftrightarrow$ *fairness*

- EFF $\cap$ ULB $\cap$ SP = EFF $\cap$ ETE $\cap$ SP =$\varnothing$ ([7]; [6])

the proof is **much** harder

the fixed priority rules are EFF $\cap$ SP $\cap$ RM $\cap$ PM $\cap$ CSY, and violates ETE, hence ANO as well

$\longrightarrow$ **from now on we only consider division rules meeting EFF and ANO**

the next two (three) rules are the main contributions of microeconomic analysis to the fair division problem

Competitive Equilibrium from Equal Incomes (CEEI): find a feasible allocation $(z^i, i \in N)$ and a price vector $p \in \mathbb{R}_+^K$ s.t. $p \cdot \omega = n$ and $z^i = \arg\max_{z: p\cdot z \leq 1} \succsim_i$ for all i

$\longrightarrow$ *fairness as equal opportunity*

efficiency is hardwired: the invisible hand at work! (and more: core stability from the equal split endowments)

existence requires convexity of preferences (as well as continuity and monotonicity);

$\longrightarrow$ *uniqueness is not guaranteed*! exceptions: the large subdomains of homogenous preferences (below), or preferences meeting gross substitutability

$\longrightarrow$ *computation is not easy in general!*

fairness properties

$\longrightarrow$ the CEEI rule meets ULB and NE (irrespective of tie-breaking)

[when agents are negligible and their preferences are connected CEEI is characterized by EFF $\cap$ NE]

$\longrightarrow$ the CEEI rule is CSY but fails RM and PM in the Arrow Debreu domain

ω-**Egalitarian Equivalent** rule (ω-EE): find an efficient allocation $(z^i, i \in N)$ and a number $\lambda, \frac{1}{n} \leq \lambda \leq 1$, such that $z^i \simeq_i \lambda\omega$ for all i

$\longrightarrow$ *fair because all have equal "benchmark welfare"*

efficiency is by construction

existence, and uniqueness of utilities hold even with non convex preferences (continuity and monotonicity are still needed)

$\longrightarrow$ *easy to compute*

the third solution is a variant of the previous one

we fix a numeraire vector $\delta \gg 0$ *in* $\mathbb{R}_+^K$

δ-**Egalitarian Equivalent** rule (δ-EE): find an efficient allocation $(z^i, i \in N)$ and a number $\lambda \geq 0$, such that $z^i \simeq_i \lambda\delta$ for all i

same interpretation, identical existence, uniqueness and computation properties

$\longrightarrow$ the ω-EE rule meets ULB and PM; it fails RM and CSY

$\longrightarrow$ the δ-EE rule is RM and PM and CSY; it fails ULB

both rules fail NE and can even lead to *Domination:* $z^i \gneq z^j$ for some agents i, j

we dismiss δ-EE rules in the sequel because a) they fail both critical single profile tests, b) the choice of δ is entirely arbitrary *an example*: **two goods X,Y, four agents with linear preferences**

utilities $5x+y, 3x+2y, 2x+3y, x+5y$; resources $\omega=(4,4)$

ω-EE allocation: $y^1=y^2=x^3=x^4=0$, and

$$5x^1=3x^2=3y^3=5y^4=24\lambda$$

$$\Rightarrow x^1=y^4=\frac{3}{2};\ x^2=y^3=\frac{5}{2}$$

exhibiting Domination

compare CEEI: $x^1=x^2=y^3=y^4=2$

an example: **two goods X,Y, and linear preferences**

fix n and $\theta, 0 \leq \theta \leq 1$, such that θn is an integer; set $\theta' = 1 - \theta$

θn agents of type "X" have utilities $2x + y$ for (x, y)

$\theta' n$ agents of type "Y" have utilities $x' + 2y'$ for (x', y')

the endowment is $\omega = (n, n)$

efficiency rules out at least one of $x' > 0$ and $y > 0$

the ω-EE allocation is symmetric (same allocation for agents of same type) and solves

$$2x + y = 3\lambda;\ x' + 2y' = 3\lambda$$

$$\theta x + \theta' x' = 1;\ \theta y + \theta' y' = 1$$

assume without loss $\theta \leq \frac{1}{2}$; the solution is $\lambda = \frac{2}{1+\theta'}$

$$x = \frac{3}{1+\theta'}, y = 0;\ x' = \frac{4\theta' - 2}{\theta'(1+\theta')}, y' = \frac{1}{\theta'}$$

the CEEI allocation hinges around the price (p_X, p_Y) normalized so that $p_X + p_Y = 1$

if $p_X \leq 2p_Y$ and $p_Y \leq 2p_X$, type X agents spend all their money to get $\frac{1}{p_X}$ units of good X, while type Y agents similarly buy $\frac{1}{p_Y}$ units of good Y; this is feasible only if $p_X = \theta, p_Y = \theta'$; so if $\frac{1}{3} \leq \theta \leq \frac{2}{3}$, the allocation is

$$x = \frac{1}{\theta}, y = 0;\ x' = 0, y' = \frac{1}{\theta'}$$

if $\theta \leq \frac{1}{3}$ the type Y agents must eat some of each good, which is only possible at the price $p_X = \frac{1}{3}, p_Y = \frac{2}{3}$ where they are indifferent about buying either good; then

$$x = 3, y = 0;\ x' = \frac{3\theta' - 2}{\theta'}, y' = \frac{1}{\theta'}$$

CEEI and ω-EE take radically different views of scarce preferences

assume θ goes from $\frac{1}{2}$ to 0, so the type X become increasingly scarce

$\longrightarrow$ the utility of both types X and types Y under ω-EE is $\frac{6}{2-\theta}$, **de**creasing from 4 to 3

$\longrightarrow$ under CEEI, while θ decreases to $\frac{1}{3}$, the utility $\frac{2}{\theta}$ of type X **in**creases from 4 to 6, the utility $\frac{2}{1-\theta}$ of type Y decreases from 4 to 3; both utilities remain flat for $\frac{1}{3} \geq \theta \geq 0$

misreporting opportunities are more severe under the ω-EE rule

ω-EE: if the number of agents is large, a type X agent i benefits by reporting utility $x + y$: the parameter λ does not change much and i gets x_i, y_i s.t. $x_i + y_i \simeq 2\lambda$ and $y_i = 0$; so $x_i \simeq 2\lambda$ improves upon $x = \frac{3}{2}\lambda$

CEEI: misreport does not pay when $\frac{1}{3} \leq \theta \leq \frac{2}{3}$ if a single message does not alter the price much; if $\theta \leq \frac{1}{3}$ a type Y agents' misreport only has a second order impact on his utility

homothetic preferences

homothetic preferences: $z \succsim z' \Rightarrow \lambda z \succsim \lambda z'$ for all $z, z' \in \mathbb{R}_+^K$ and all $\lambda > 0$

representable by utility u homogenous of degree one

Theorem *(Eisenberg, Chipman, Moore)*: *under homothetic preferences, the CEEI allocation maximizes the Nash CUF* $\sum_N \ln\{u_i(z^i)\}$ *over all feasible allocations* $(z^i, i \in N)$

the proof is elegantly simple, see Chapter 14 in [26]

$\Rightarrow$ the CEEI solution is unique utility-wise (and even allocation-wise if the functions u_i are log-concave) moreover if the CEEI rule is RM on some homogenous subdomain, it is also PM on that domain

$\Longrightarrow$ it is easy to compute

we look at three subdomains of homothetic preferences, useful in applications because each preference is described by a vector $\beta \in \mathbb{R}_{+}^{K}$ normalized by $\sum_A \beta_a = 1$

- Cobb-Douglas: $u(z) = \sum_A \beta_a \ln(z_a)$

- linear: $u(z) = \sum_A \beta_a z_a$

- Leontief: $u(z) = \min_a\{\frac{z_a}{\beta_a}\}$ (where $\beta \gg 0$)

linear preferences have maximal substitutability, Leontief ones have maximal complementarity, with Cobb-Douglas preferences somewhere in between

Cobb-Douglas preferences

the CEEI allocation is computed in closed form

$$\text{price } p_a = \frac{\beta_a^N}{\omega_a};\ z^i = \arg\max_{z:p\cdot z\leq 1}\{\sum_A \beta_a^i \ln(z_a^i)\} = (\frac{\beta_a^i}{\beta_a^N}\omega_a, a \in A)$$

implying at once that the CEEI rule is RM, hence PM as well

the ω-EE allocation cannot be computed in closed form; its computational complexity appears to be high

the ω-EE rule is RM, and PM as always

$\rightarrow$ the two solutions have very similar properties (CSY is the only exception), in particular neither is SP on the Cobb Douglas domain

linear preferences

Proposition *the CEEI rule is RM, hence PM as well*

the proof is not simple, and neither is the computation of the solution

the ω-EE rule violates RM in the linear domain (see [1])

neither solution is SP on the linear domain ([8])

linear + dichotomous preferences

agent i likes the commodities in A_i as equally good, others are equally bad

canonical utility $u_i(z^i) = z^i_{A_i}$

assume $\cup_N A_i = A$; notation $A_S = \cup_S A_i$

efficiency: all goods are eaten and i consumes only goods in A_i

utility profile $(u_i, i \in N)$ is feasible *iff* $u_N = \omega_N$ and $u_S \leq \omega_{A_S}$ for all $S \subset N$

Proposition: *the CEEI utility profile is the Lorenz dominant feasible profile; the CEEI rule is RM, PM, and (Group)SP*

$\longrightarrow$ the ω-EE utility profile becomes similarly the Lorenz dominant feasible profile of relative utilities $(\frac{u_i}{\omega_{A_i}}, i \in N)$; it maximizes the weighted Nash CUF $\sum_N \omega_{A_i} \ln\{u_i\}$ in the feasible set

$\longrightarrow$ the ω-EE rule is RM and PM, but not SP

cake division

Ω a compact set in $\mathbb{R}^L$: the cake

agent i's utility for a (Lebesgue-measurable) piece of cake A: $\int_A u_i(x)dx$ (or simply $\int_A u_i$)

the density u_i is strictly positive and continuous on Ω, and normalized as $\int_\Omega u_i = 1$

agent i's share is A_i, where $\{A_i, i \in N\}$ is a partition of Ω

a partition is efficient if and only if

$$\min_{A_i} \frac{u_i}{u_j} \geq \max_{A_j} \frac{u_i}{u_j} \text{ for all } i, j$$

hence $\frac{u_i}{u_j}$ is constant on any contact line of A_i and A_j

consequence of additivity of utilities: NE $\Rightarrow$ ULB

ω-EE allocation: each agent receives the same fraction of total utility (normalized to 1), therefore $\int_{A_i} u_i = \int_{A_j} u_j$ for all i, j

the CEEI partition maximizes the Nash CUF; the KT conditions read

$$\frac{u_i(x)}{\int_{A_i} u_i} \geq \frac{u_j(x)}{\int_{A_j} u_j} \text{ for all } i \text{ and all } x \in A_i$$

write i's net utility $U_i = \int_{A_i} u_i$; the KT conditions amount to

$$\min_{A_i} \frac{u_i}{u_j} \geq \frac{U_i}{U_j} \geq \max_{A_j} \frac{u_i}{u_j} \text{ for all } i, j$$

the price is simply $p(x) = \frac{u_i(x)}{\int_{A_i} u_i}$ for $x \in A_i$

cake cutting as a limit case of the linear preferences AD model

if each density u_i takes only finitely many distinct values (therefore discontinuous), cake division is an instance of the AD model with linear preferences

a limit argument carries the properties of the linear model to cake division:

$\Rightarrow$ CEEI is RM and PM, ω-EE is PM (see a direct proof in [23]) *cake cutting* is the subject of a large mathematical literature: e.g., [13],[12], see [14] for a survey

and (recently) algorithmic literature: [**?**], [10], [2]

its own terminology

ULB $\leftrightarrow$ proportional: $\int_{A_i} u_i \geq \frac{1}{n}$

EE $\leftrightarrow$ equitable: $\int_{A_i} u_i = \int_{A_j} u_j$ for all i, j

goal: find simple "cutting" or "knife-stopping" algorithms to implement a non envious allocation, or an equitable allocation

Leontief preferences

the definition of the ω-EE allocation is altered to rule out waste, then it is computed in almost closed form

$$\{z^i = \mu_i \beta^i \text{ and } \mu_i = \lambda u_i(\omega)\} \Rightarrow \lambda\{\sum_N u_j(\omega)\beta_a^j\} \leq \omega_a$$

the optimal λ is $\min_a \frac{\omega_a}{\sum_N u_j(\omega)\beta_a^j}$ therefore

$$z^i = \min_a \frac{u_i(\omega)\omega_a}{\sum_N u_j(\omega)\beta_a^j}\beta^i, \text{ and } u_i(z^i) = \min_a \frac{u_i(\omega)\omega_a}{\sum_N u_j(\omega)\beta_a^j}$$

Theorem ([4],[5]): *the non wasteful ω-EE rule is GSP, NE, RM, PM, and CSY*

the only missing axiom is ULB

it is possible to define rules meeting GSP, ULB, and PM

$\longrightarrow$ compare CEEI: not SP and neither RM nor PM

many more mechanisms meet the axioms in the theorem; they respect the spirit of ω-EE to equalize utilities along a benchmark ([5])

one non disposable commodity

a variant of the AD model: satiated preferences, no free disposal

examples: sharing a workload, a risky investment, a fixed amount of a fixed price commodity

$\omega \in \mathbb{R}_+$: amount of resource to divide (infinitely divisible)

$\succsim_i$: agent i's preferences over $[0, \omega]$: single-peaked, i.e., unique maximum π^i, strictly increasing (decreasing) before (after) π^i

$\longrightarrow$ feasible allocation $(z^i \in \mathbb{R}_+, i \in N)$,$\sum_N z^i = \omega$

$\longrightarrow$ efficient allocation:

if $\sum_N \pi^i \geq \omega$ then $z^i \leq \pi^i$ (excess demand)

if $\sum_N \pi^i \leq \omega$ then $z^i \geq \pi^i$ (excess supply) the uniform solution

$$\text{if } \sum_N \pi^i \geq \omega \text{ then } z^i = \min\{\lambda, \pi^i\} \text{ where } \sum_N \min\{\lambda, \pi^i\} = \omega$$

$$\text{if } \sum_N \pi^i \leq \omega \text{ then } z^i = \max\{\lambda, \pi^i\} \text{ where } \sum_N \max\{\lambda, \pi^i\} = \omega$$

CEEI-like interpretation: if excess demand, price 1 and budget λ, disposable; if excess supply, price 1, unbounded budget, must spend at least λ

Resource/Population Monotonicity need adapting: more resources/ fewer agents is good news if excess demand, bad news if excess supply

*Resource Monotonicity** (RM*): more resources means either weakly good news for everyone, or weakly bad news for everyone

*Population Monotonicity** (PM*): one more agent means either weakly good news for all current agents, or weakly bad news for all

Theorem ([24],[22]) *the uniform solution is SP, NE, RM*, PM*, and CSY; it is characterized by the combination of EFF, ETE, and SP*

$\longrightarrow$ the uniform rule is the compelling fair division rule

$\longrightarrow$ the division in proportion to peaks plays no special role because agents are responsible for their preferences (compare with the claims problem, where a claim π^i is an objective "right", and proportional division is a major player)

assignment

a variant of the AD model with several comparable commodities (similar jobs), and fixed individual total shares of commodity

special case: random assignment of indivisible goods (one item per agent)

$N \ni i$: agents, $|N| = n$

$A \ni a$: goods, $|A| = K$

$\omega \in \mathbb{R}_+^K$: resources to divide (infinitely divisible)

agent i has a quota q^i

$\longrightarrow$ an allocation $(z^i, i \in N)$ is feasible if

$$z^i \in \mathbb{R}^K_+; \; \sum_N z^i = \omega; \; \sum_A z^i_a = q^i$$

because we focus on anonymous division rules, we assume $q^i = \frac{1}{n} \sum_A \omega_a$

the random assignment model: $|A| = n, \omega_a = 1$ for all a, z_a^i is the probability that i gets object a

Birkhof's theorem $\Rightarrow$ {random assignment of the indivisible goods} $\Leftrightarrow$ {deterministic assignment of the divisible goods}

we discuss three preference domains

dichotomous preferences

the particular case of ordinal preferences where commodities (objects) are viewed as good or bad (two indifference classes) $\Rightarrow$ the relation $\succsim_i^{sd}$ is complete, represented by the canonical utility $u_i(z) = \sum_{a \text{ is good for } i} z_a$

Proposition ([21]): *CEEI allocations and PS allocations (leximin definition) coincide; their unique utility profile is Lorenz dominant in the feasible set; the corresponding rule is (are) (group) SP*

$\longrightarrow$ similar to the case of manna with linear dichotomous preferences

linear preferences (vonNeumanMorgenstern utilities)

CEEI rule: find a price $p \in \mathbb{R}_+^K$ and a feasible $(z^i, i \in N)$ such that

$$z^i \in \arg \max_{p \cdot z \leq 1, z_A = q^i} \{\beta \cdot z\} \text{ for all } i$$

ω-EE rule: find a positive number λ and an efficient feasible $(z^i, i \in N)$ such that $\beta \cdot z^i = \lambda(\beta \cdot \omega)$ for all i

the Eisenberg Chipman Moore theorem does not hold any more: the CEEI solution does not maximize the Nash product, and is typically not unique utility-wise and allocation-wise

$\longrightarrow$ all easy properties are preserved: CEEI meets ULB, NE, CSY

$\longrightarrow$ ω-EE meets ULB, PM, but generates Domination and fails CSY

$\longrightarrow$ neither solution is SP

Open question: what single-valued selection from the CEEI rule can we recommend?

ordinal preferences

in practical instances of the random assignment problem (school choice, campus rooms, time slots, similar jobs), we can only elicit from each agent i her ordinal ranking $\succ_i$ of the various goods; this yields a partial ordering $\succ_i^{sd}$ of her allocations

if $a \succ_i b \succ_i c \succ_i \cdots$

$z \succ_i^{sd} z' \overset{def}{\Leftrightarrow} z_a \geq z'_a,\ z_a + z_b \geq z'_a + z'_b,$ etc.., with at least one strict inequality

(sd : stochastic dominance for the probabilistic interpretation; otherwise Lorenz dominance) the feasible allocation $(z^i, i \in N)$ is *ordinally efficient* iff there is no feasible allocation $(z'^i, i \in N)$ such that $z \succsim_i^{sd} z'$ for all i, with at least one strict relation

$\longrightarrow$ this notion is stronger than ex post efficiency, and weaker than ex ante efficiency the ω-EE allocation cannot be adapted in the absence of a complete preference relation; same remark for the Nash CUF

the **Probabilistic Serial** (PS) allocation has two equivalent definitions:

- eating algorithm: agents eat at the same speed from their best commodity among those not yet exhausted

- leximin optimum of the Lorenz profile (Bogomolnaia)

- it can also be interpreted as a version of CEEI (Kesten) *leximin definition of PS*

the Lorenz curve of z at $\succ_i$ is $Lc(z, \succ_i) = (z_a, z_a + z_b, z_a + z_b + z_c, \cdots)$ where top $= a \succ_i b \succ_i c \succ_i \cdots$

the *Lorenz profile* $Lc(z, \succ)$ concatenates the Lorenz curves $Lc(z^i, \succ_i)$ at the allocation $z = (z^i, i \in N)$ and preference profile $\succ = (\succ_i, i \in N)$

Proposition: the Lorenz profile $Lc(z^*, \succ)$ of the PS allocation z^* is leximin optimal:

$$Lc(z^*, \succ) \succsim_{lxmin} Lc(z, \succ) \text{ for all feasible allocations } z$$

this definition holds even if preferences exhibit some indifferences; the eating algorithm is harder to adjust to indifferences (Katta Sethuraman) example: random assignment with 3 agents and 3 objects

$$\begin{array}{ccccc} a & \succ_1 & b & \succ_1 & c \\ a & \succ_2 & c & \succ_2 & b \\ b & \succ_3 & & a, & c \end{array}$$

$$PS = \begin{array}{ccc} a & b & c \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{array}$$

Resource/Population Monotonicity: more resources, or one agent less means a larger quota for everyone

if $z_A \geq z'_A$ we say that z is sd preferred to z' iff $z_{/z'} \succsim_i^{sd} z'$, where $z_{/z'}$ collects the z'_A best units of commodities for agent i; we still write $z \succsim_i^{sd} z'$

the definition of RM, PM is then the same

Strategyproofness: if i gets z^i by telling the truth $\succsim_i$, and z'^i by telling a lie, sd-SP requires $z \succsim_i^{sd} z'$, whereas weak-SP only asks $\rceil z' \succsim_i^{sd} z$

Theorem ([20])

i) *the PS meets ordinal-EFF; (sd) ULB, NE, RM, PM; and weak-SP*

ii) *for* $n \geq 4$, *there is no assignment rule meeting ordinal-EFF, ETE, and sd-SP* the **Random Priority** assignment is simply the average of the fixed priority assignments (first in line takes his best q^i units, next one takes his best q^i in what is left, etc..); it is a popular method, easier to implement than PS, but much harder to "compute"

RP has stronger incentives properties than PS

$\longrightarrow$ *the RP rule meets sd–ULB, sd-SP, and weak-NE*

back to the example

$$RP = \begin{matrix} a & b & c \\ \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{5}{6} & \frac{1}{6} \end{matrix} \qquad PS = \begin{matrix} a & b & c \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{3}{4} & \frac{1}{4} \end{matrix}$$

RP has weaker efficiency properties than PS: *not ordinally efficient*

$$a \succ_1 b \succ_1 c \succ_1 d$$
$$a \succ_1 b \succ_1 c \succ_1 d$$
$$b \succ_1 a \succ_1 d \succ_1 c$$
$$b \succ_1 a \succ_1 d \succ_1 c$$

scheduling example with deadline (opting out)

4 agents with deadlines respectively $t = 1, 2, 3, 4$

$$RP = \begin{matrix} \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{3}{8} & \frac{1}{24} \end{matrix} \qquad PS = \begin{matrix} \frac{1}{4} & 0 & 0 & 0 \\ \frac{1}{4} & \frac{1}{3} & 0 & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{5}{12} & 0 \\ \frac{1}{4} & \frac{1}{3} & \frac{5}{12} & 0 \end{matrix}$$

$\longrightarrow$ PS stochastichally dominates RP (not a general feature !)

$\longrightarrow$ PS and RP are assymptotically equivalent

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