Picking Sequences

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A fair division problem...

- 4 candies of different flavours: mint, strawberry, lemon, licorice.
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2. ABAB?
3. ABBB?
4. ABBA?
5. and what if we have 5 candies?

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4. ABBA?
5. and what if we have 5 candies? or 17 candies and 5 agents?

Picking sequences

- A set $\mathcal{O}$ of p *items* (or *objects*) $\{o_1, \ldots, o_p\}$
- A set $\mathcal{N}$ of n *agents* $\{1, \ldots, n\}$
- Each agent i has a (private) *ranking* $\succ_i$ over $\mathcal{O}$
- $\pi = (\pi(1), \ldots, \pi(p))$ with $\pi(i) \in \mathcal{N}$ for all j: *picking sequence*
- At each step, the designated agent takes one item among those that remain.

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$$\pi = ABBAA$$

$$\rhd_A: o_1 \quad o_2 \quad o_3 \quad o_4 \quad o_5$$
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Final allocation: $[o_1 o_4 o_5 | o_2 o_3]$

Part I Finding optimal picking sequences

Part II Manipulating picking sequences

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1. **Scoring:** $g : \{1, \ldots, p\} \mapsto \mathbb{R}^+$ *scoring function* (common to all agents) mapping rank to values: $u(o) = g(rank(o, \succ_i))$.
2. **Additivity:** for each $S \subseteq \mathcal{O}$ and $i \in \mathcal{N}$,

$$u_i(S) = \sum_{o \in S} u_i(o) = \sum_{o \in S} g(rank(o, \succ_i))$$

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Back to the example

Example

5 items, 3 agents, $\pi = 12332\ldots$

- $1 : \mathbf{o_1} \succ o_2 \succ o_3 \succ o_4 \succ o_5$
- $2 : \mathbf{o_4} \succ \mathbf{o_2} \succ o_5 \succ o_1 \succ o_3$
- $3 : o_1 \succ \mathbf{o_3} \succ \mathbf{o_5} \succ o_4 \succ o_2$

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- **Borda:** $u_1(\pi) = 5$; $u_2(\pi) = 5 + 4 = 9$; $u_3(\pi) = 4 + 3 = 7$.
- **lexicographic:** $u_1(\pi) = 16$; $u_2(\pi) = 24$; $u_3(\pi) = 12$.
- **QI:** $u_1(\pi) = 1 + 4\varepsilon$; $u_2(\pi) = 2 + 7\varepsilon$; $u_3(\pi) = 2 + 5\varepsilon$.

Social welfare

We use a *collective utility function* to aggregate the individual utilities.

Two well-known functions:

- **utilitarian:** $F(u_1, \ldots, u_n) = \sum_{i=1,\ldots,n} u_i$
- **egalitarian:** $F(u_1, \ldots, u_n) = \min_{i=1,\ldots,n} u_i$

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The procedure is elicitation-free. . .

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The central authority has a prior probability on the preference profile:

- **Full independence:** each profile $R = \langle \succ_1, \ldots, \succ_n \rangle$ is equally probable (uniform distribution over profiles)
- **Full correlation:** all agents have the same preferences: $R = \langle \succ, \ldots, \succ \rangle$.

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Expected individual and collective utilities:

$$\overline{u(i,\pi)} = \sum_R Pr(R) u_i(\pi, R).$$

$$\overline{sw_F(\pi)} = F(\overline{u(1,\pi)}, \ldots, \overline{u(n,\pi)}).$$

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Summary

- **Instance:**
  - a number of agents n
  - a number of items p
  - a scoring function g
  - a prior (i.e., a correlation assumption) $C \in \{FC, FI\}$
  - an aggregation function F
- **Question:**
  - *FInd the sequence π maximizing $\overline{sw_F(\pi)}$, under prior C*

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(Reduction from PARTITION)

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What about...

- ...lexicographic scoring ?
- ...quasi-indifference scoring ?
- ...Borda scoring ?

Full correlation + Lexicographic scoring + Egalitarian

$\succ_i$	o_6	$\succ$	o_1	$\succ$	o_4	$\succ$	o_5	$\succ$	o_2	$\succ$	o_3
lexicographic	32	$>>$	16	$>>$	8	$>>$	4	$>>$	2	$>>$	1

Egalitarian CUF (min)

Optimal sequences:

$$\sigma(1)\sigma(2)\ldots\sigma(n-1)\sigma(n)^{p-n+1}$$

where σ is a permutation of $\{1,\ldots,n\}$

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$\pi = 123333$

- $\overline{u(1,\pi)} = 32$
- $\overline{u(2,\pi)} = 16$
- $\overline{u(3,\pi)} = 8+4+2+1 = 15$

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- polynomial number of sums of values for each agent
- total number of vector of values $\leq \left(\frac{p(p+1)}{2}\right)^n$
- can be solved by dynamic programming
- polynomial in p (but not in n).

Full correlation + QI scoring + Egalitarian

$\succ_i$	o_6	o_1	o_4	o_5	o_2	o_3
Quasi-indifference	$1+5\varepsilon$	$1+4\varepsilon$	$1+3\varepsilon$	$1+2\varepsilon$	$1+\varepsilon$	1

- Comes down to solving the Borda case!
- Intuition:
  - let $m = \lfloor \frac{p}{n} \rfloor$ and $q = p - nm$
  - for instance, $p = 10$, $n = 4 \Rightarrow q = 2$
  - Optimal sequences: $\pi = \underbrace{1122}_{n-q \text{ agents}} \overbrace{333444}^{q \text{ agents}}$ and

    $\pi' = \underbrace{1221}_{n-q \text{ agents}} \overbrace{333444}^{q \text{ agents}}$
  - The last q agents are OK $\rightarrow u \geq m+1$
  - The first $n-q$ agents: $u = m + x \cdot \varepsilon$ ($x \rightarrow$ Borda)

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General case: precise complexity of the problems of:

- computing the expected collective utility of a sequence
- computing the best sequence

are still **open questions** (NP-hardness conjectured).

Many items

- When $p \to +\infty$, allocations of the form $\sigma_1\sigma_2\ldots\sigma_k\theta$, where $\sigma_1,\ldots,\sigma_k$, are permutations of $\{1,\ldots,n\}$ (and $p = kn+q$), tend to be optimal.
  Example: 123 123 321 231 132 321 123 . . .

Some examples

Assumptions: Full independence, egalitarian, Borda scoring.

p	$n = 2$	$n = 3$
4	1221	1233
5	11222	12332
6	121221	123321
8	12212112	11332232
10	1221121221	1231223133
12	121212122121	?

http://recherche.noiraudes.net/en/sequences.php

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p	$n = 2$	$n = 3$
4	1212	1231
5	12121	12312
6	121212	123123
8	12121212	12312312
10	1212121212	1231231231
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Picking sequences: manipulation

- $m = 5$; $\pi = ABBAA$;
- If both players act sincerely:

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  final allocation: $[o_1 o_2 o_4 | o_3 o_5]$
  if A has separable preferences then she prefers $o_1 o_2 o_4$ to $o_1 o_4 o_5$
- Side question: which picking sequences are strategyproof for all agents?
- Side question: is there a simple preference domain under which all picking sequences are strategyproof?

Manipulating picking sequences: the two-agent case

2 agents: *A* manipulator, *B* non-manipulator.
Question: what is the best set of items *A* can obtain?

- $S \subseteq O$ is *achievable* (against *B*'s strategy) if *A* has a picking strategy that leads him to get a subset of items containing *S*.
- A strategy for *A* is an *optimal manipulation* if *A* obtains the best subset of items against *B*'s strategy.

- Previous work: Kohler and Chandrasekaran (1971) show that for the alternation sequence *ABABAB*..., an optimal manipulation can be found in polynomial time.

Manipulating picking sequences: the two-agent case

- Two agents $\{A, B\}$, with A as the manipulator.
- (for now) Assumption 1: *A has additive preferences*, represented by a utility function over single items $u : \mathcal{O} \rightarrow \mathbb{R}^+$, *with no ties between single items*: $o \neq o'$ implies $u(o) \neq u(o')$.
- $\rhd_A$ well-defined linear order over items: $o \rhd o'$ if $u_A(o) > u_A(o')$

Manipulating picking sequences: the two-agent case

- Two agents $\{A, B\}$, with A as the manipulator.
- Assumption 1: *A has additive preferences*
- Assumption 2: *A has a complete knowledge of B's picking strategy*
- Assumption 3: *B's picking strategy is deterministic and simple*: B always picks the preferred item, with respect some preference relation $\rhd_B$, among those that remain available.
  - $\rhd_B$ may be B's sincere preference of B, but need not to be.
  - similar assumptions in voting manipulation.

Manipulating picking sequences: the two-agent case

A greedy algorithm:

- **Input**: picking sequence π, A's preferences $\rhd_A$, B's behaviour $\rhd_B$
- **Output**: the best achievable subset for A
- **Begin**
- $S \leftarrow \emptyset$;
- $j \leftarrow 1$;
- **for** $k \leftarrow 1$ to the number of picking stages for A
  - find the smallest $i \geq j$ such that $S \cup \{i\}$ is achievable;
  - $S := S \cup \{i\}$
  - $j := i$
- **Return** S;
- it can be checked in polynomial time whether a given set is achievable
- therefore the algorithm works in polynomial time.

Manipulating picking sequences: the two-agent case

Checking whether a set of items is achievable

- S is achievable if and only if the *standard picking strategy*, in which A picks items in S according to their increasing ranking in $\rhd_B$, is successful.
- $m = 12$; $\pi = ABABABABABAB$;

$\rhd_A$:	1	2	3	4	5	6	7	8	9	10	11	12
$\rhd_B$:	3	2	6	5	4	10	8	11	1	9	7	12

Manipulating picking sequences: the two-agent case

Checking whether a set of items is achievable

- S is achievable if and only if the *standard picking strategy*, in which A picks items in S according to their increasing ranking in $\rhd_B$, is successful.
- $m = 12$; $\pi = ABABABABABABAB$;
- 1245 is achievable:
- $2 \rhd_B 5 \rhd_B 4 \rhd_B 1$

$\rhd_A$:	1	2	3	4	5	6	7	8	9	10	11	12
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$\rhd_B$:	**3**	2	6	5	4	10	8	11	1	9	7	12

Manipulating picking sequences: the two-agent case

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Manipulating picking sequences: the two-agent case

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- S is achievable if and only if the *standard picking strategy*, in which A picks items in S according to their increasing ranking in $\triangleright_B$, is successful.
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- 1245 is achievable:
- $2 \triangleright_B 5 \triangleright_B 4 \triangleright_B 1$

$\triangleright_A$:	1	2	3	4	5	6	7	8	9	10	11	12
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Manipulating picking sequences: the two-agent case

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Manipulating picking sequences: the two-agent case

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$\rhd_B$:	3	2	6	5	4	10	8	11	1	9	7	12

Manipulating picking sequences: the two-agent case

Checking whether a set of items is achievable

- *S* is achievable if and only if the *standard picking strategy*, in which *A* picks items in *S* according to their increasing ranking in $\rhd_B$, is successful.
- $m = 12$; $\pi = ABABABABABABAB$;
- 1245 is achievable:
- $2 \rhd_B 5 \rhd_B 4 \rhd_B 1$

$\rhd_A$:	1	2	3	4	5	6	7	8	9	10	11	12
$\rhd_B$:	3	2	6	5	4	10	8	11	1	9	7	**12**

Manipulating picking sequences: the two-agent case

Checking whether a set of items is achievable

- S is achievable if and only if picking items in S first and according to their increasing ranking in $\rhd_B$, is successful.
- $m = 12$; $\pi = ABABABABABAB$;
- 1245 is achievable:
  - $2 \rhd_B 5 \rhd_B 4 \rhd_B 1$

$\rhd_A$:	1	2	3	4	5	6	7	8	9	10	11	12
$\rhd_B$:	3	2	6	5	4	10	8	11	1	9	7	12

- 123 is not achievable:
  - $3 \rhd_B 2 \rhd_B 1$

$\rhd_A$:	1	2	3	4	5	6	7	8	9	10	11	12
$\rhd_B$:	3	2	6	5	4	10	8	11	1	9	7	12

Manipulating picking sequences: the two-agent case

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$m = 12$; $\pi = ABABABABABABAB$;

$\rhd_A$:	1	2	3	4	5	6	7	8	9	10	11	12
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Sincere picking stategy for A (and B) leads to A getting 124789.
Fiding an optimal manipulation:

- 1 is achievable; $S = \{1\}$;

Manipulating picking sequences: the two-agent case

$m = 12$; $\pi = ABABABABABABAB$;

$\rhd_A$:	1	2	3	4	5	6	7	8	9	10	11	12
$\rhd_B$:	3	2	6	5	4	10	8	11	1	9	7	12

Sincere picking stategy for *A* (and *B*) leads to *A* getting 124789.

Fiding an optimal manipulation:

- 1 is achievable; $S = \{1\}$;
- 1 2 is achievable; $S = \{1, 2\}$;

Manipulating picking sequences: the two-agent case

$m = 12$; $\pi = ABABABABABABAB$;

$\rhd_A$:	1	2	3	4	5	6	7	8	9	10	11	12
$\rhd_B$:	3	2	6	5	4	10	8	11	1	9	7	12

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- 1 2 is achievable; $S = \{1,2\}$;
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Manipulating picking sequences: the two-agent case

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- 1 2 is achievable; $S = \{1,2\}$;
- 1 2 3 is not achievable; $S = \{1,2\}$;
- 1 2 4 is achievable; $S = \{1,2,4\}$;

Manipulating picking sequences: the two-agent case

$m = 12$; $\pi = ABABABABABABAB$;

$\rhd_A$:	1	2	3	4	5	6	7	8	9	10	11	12
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- 1 2 3 is not achievable; $S = \{1,2\}$;
- 1 2 4 is achievable; $S = \{1,2,4\}$;
- 1 2 4 5 is achievable; $S = \{1,2,4,5\}$;

Manipulating picking sequences: the two-agent case

$m = 12$; $\pi = ABABABABABABAB$;

$\rhd_A$:	1	2	3	4	5	6	7	8	9	10	11	12
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- 1 2 4 is achievable; $S = \{1,2,4\}$;
- 1 2 4 5 is achievable; $S = \{1,2,4,5\}$;
- 1 2 4 5 6 is not achievable; $S = \{1,2,4,5\}$;

Manipulating picking sequences: the two-agent case

$m = 12$; $\pi = ABABABABABABAB$;

$\rhd_A$:	1	2	3	4	5	6	7	8	9	10	11	12
$\rhd_B$:	3	2	6	5	4	10	8	11	1	9	7	12

Sincere picking stategy for A (and B) leads to A getting 124789.

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- 1 2 4 5 is achievable; $S = \{1,2,4,5\}$;
- 1 2 4 5 6 is not achievable; $S = \{1,2,4,5\}$;
- 1 2 4 5 7 is achievable; $S = \{1,2,4,5,7\}$;

Manipulating picking sequences: the two-agent case

$m = 12$; $\pi = ABABABABABABAB$;

$\triangleright_A$:	1	2	3	4	5	6	7	8	9	10	11	12
$\triangleright_B$:	3	2	6	5	4	10	8	11	1	9	7	12

Sincere picking stategy for A (and B) leads to A getting 124789.

Fiding an optimal manipulation:

- 1 is achievable; $S = \{1\}$;
- 1 2 is achievable; $S = \{1,2\}$;
- 1 2 3 is not achievable; $S = \{1,2\}$;
- 1 2 4 is achievable; $S = \{1,2,4\}$;
- 1 2 4 5 is achievable; $S = \{1,2,4,5\}$;
- 1 2 4 5 6 is not achievable; $S = \{1,2,4,5\}$;
- 1 2 4 5 7 is achievable; $S = \{1,2,4,5,7\}$;
- 1 2 4 5 7 8 is achievable; $S = \{1,2,4,5,7,8\}$; stop and return S.

Manipulating picking sequences: the two-agent case

Lemma:

- Let S and S' two achievable sets of items by A.

  $S = 124789$; $S' = 1245$
- Let $i = \min((S' \setminus S) \cup (S \setminus S'))$ and assume wlog that $i \in S'$.

  $1 \rhd_A 2 \rhd_A 3 \rhd_A 4 \rhd_A 5 \rhd_A \ldots \Rightarrow i = 5$
- Let j be B's preferred item in $S \setminus S'$.

  $S \setminus S' = 789$; $8 \rhd_B 9 \rhd_B 7 \Rightarrow j = 8$
- Then $S[i \leftrightarrow j] = (S \cup \{i\}) \setminus \{j\}$ is achievable.

  $S[5 \leftrightarrow 8] = 124579$ is achievable

Consequences:

- the algorithm returns the best achievable subset for A.
- an optimal manipulation can be computed in polynomial time.
- the optimal manipulations for A are the same for any utility function u_A compatible with $\rhd_A$.

Manipulating picking sequences: the two-agent case

We now relax assumption

3' A has no ties between single items

- we have now a weak order $\trianglerighteq_A$ over items

An optimal achievable set of items for A can be computed as follows:

- let $\triangleright'_A$ be the linear order on $\mathcal{O}$ refining $\trianglerighteq_A$ and defined by:

$$o \triangleright'_A o' \text{ iff } o \triangleright_A o' \text{ or } (o \sim_A o' \text{ and } o \triangleright_B o')$$

- find an optimal manipulation with $\triangleright'_A$.
- then the (unique) optimal achievable subset is a (non necessarily unique) optimal achievable subset for $(\mathcal{O}, u_A, \triangleright_B, \pi)$.

Consequences:

- (again) optimal manipulation can be computed in polynomial time.
- (again) optimal manipulations for A are the same for any utility function u_A compatible with $\trianglerighteq_A$.

One manipulator, at least two nonmanipulators

- One manipulator A, $n-1$ non-manipulators $B_1, \ldots, B_{n-1}$
- A knows $\succ_{B_1}, \ldots, \succ_{B_1}$

For each manipulation problem

$$P = ((A, B_1, \ldots B_n), S, \pi, (\succ_{B_1}, \ldots, \succ_{B_n}))$$

there exists a two-agent problem

$$P' = ((A, B), S, \pi', \succ_B)$$

such that S is achievable by A in P *iff* S is achievable by A in P'.

One manipulator, at least two nonmanipulators

- $\pi = ABCABCABCABC$;
- A manipulator; B, C non-manipulators;
- Can A obtain 1 2 3 4?
- $\succ_A =$ 1 2 3 4 5 6 7 8 9 10 11 12
- $\succ_B =$ 7 8 1 10 12 3 2 11 5 4 9 6
- $\succ_C =$ 8 9 2 1 6 5 4 12 11 10 7 3
- A knows $\succ_B$ and $\succ_C$

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- $\pi' = ADDADDADDADD$
- $\succ_D =$ 7 8 1 10 9 12 2 6 3 11 5 4

One manipulator, at least two nonmanipulators

1. The problem of finding a strategy to get a set of items against several nonmanipulators can be reduced to finding a strategy to get a set of items against one nonmanipulator
2. For a manipulator with additive preferences, and two agents, an optimal set of items can be found in polynomial time.

Do 1 and 2 entail

3. For a manipulator with additive preferences, and several nonmanipulators, an optimal set of items can be found in polynomial time ?

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Do 1 and 2 entail

3. For a manipulator with additive preferences, and several nonmanipulators, an optimal set of items can be found in polynomial time ?

- NO! This problem is NP-hard (Aziz, Bouveret, Lang and Mackenzie, 2016).
- Reduction from 3-SATISFIABILITY

Several manipulators: coalitional manipulation

Three notions of manipulation:

- *simple*: combination of picking strategies whose outcome Pareto-dominates (for the manipulating coalition) the outcome of the sincere picking strategy.
- *with post-allocation item trade*: also based on Pareto-dominance but agents are allowed to trade items after the allocation has been done.
- *with post-allocation item trade + money transfer*: the manipulators' preferences are represented by transferable utilities (= money); agents are allowed both to trade items *and* transfer money after the allocation has been done.

Several manipulators: coalitional manipulation

Simple coalitional manipulation: no post-allocation trade is allowed.

- A and B manipulators, C non-manipulator.
- $\pi = ABCABC$.
- $\rhd_A$: 1 2 5 4 3 6; $\rhd_B$: 1 3 5 2 4 6; $\rhd_C$: 2 3 4 1 5 6
- Sincere picking leads to $[15|34|26]$.
- A and B manipulating alone cannot do better
- If they cooperate, they both can do better:
  - A starts by picking 2
  - B picks 3
  - C picks 4
  - A picks 1
  - B picks 5 and finally C picks 6.
  - final allocation $[12|35|46]$
  - Pareto-dominates $[15|34|26]$.
- A and B must communicate beforehand and trust each other!

Several manipulators: coalitional manipulation

Post-allocation exchange of goods is allowed. Monetary transfers are not allowed

- A and B manipulators, C non-manipulator.
- $\pi = ABCABCABC$.
- $\rhd_A$: 1 2 3 4 5 6 7 8 9; $\rhd_B$: 8 9 3 4 5 6 7 1 2; $\rhd_C$: 1 2 3 8 9 7 4 5 6
- Sincere picking leads to $[1\,3\,4 | 5\,8\,9 | 2\,6\,7]$.
- A and B manipulating without exchanging goods cannot do better.
- If they cooperate and are allowed to exchange goods, then
  - A picks 1, B picks 2, C picks 3
  - A picks 8, B picks 9, C picks 7
  - A picks 4, B picks 5 and C picks 6,
  - resulting allocation $[1\,4\,8 | 2\,5\,9 | 3\,6\,7|$.
  - then A and B exchange 2 and 8
  - new allocation: $[1\,2\,4 | 5\,8\,9 | 3\,6\,7|$
  - Pareto-dominates $[1\,3\,4 | 5\,8\,9 | 3\,6\,7|$ for $\{A, B\}$.

Several manipulators: coalitional manipulation

Post-allocation exchange of goods and monetary transfers allowed.

- A and B manipulators, C non-manipulator.
- $\pi = ABCABCABC$
- $\rhd_A$: 1 2 3 4 5 6 7 8 9; $\rhd_B$: 3 4 5 9 1 6 7 8 2; $\rhd_C$: 1 2 3 8 9 7 4 5 9
- A and B have additive preferences corresponding to the amount of money they are willing to pay to get the items:
  - $u_A(1) = 14$; $u_A(2) = 13$; $u_A(3) = 12$; $u_A(4) = 11$; $u_A(5) = 10$; $u_A(6), u_A(7)... \leq 5$;
  - $u_B(9) = 10$; $u_B(8) = 9$; $u_B(7) = 8$; $u_B(6) = 7$; $u_B(5) = 6$
- sincere picking leads to $[1\,2\,5 \,|\, 7\,8\,9 \,|\, 3\,4\,6]$
- A and B can cooperate and get $[1\,4\,7 \,|\, 2\,5\,9 \,|\, 3\,6\,8]$
- then B gives 2 and 5 to A, A gives 7 to B together with some amount of money.
- Both are strictly better off.
- Needs transferable utilities.

Several manipulators: coalitional manipulation

- *with money transfers and exchanges*:
  (a) in the optimal final allocation (after the exchanges), each item will be assigned to the agent who gives it the highest utility
  (b) the optimal joint picking strategy is the one that maximizes the utilitarian social welfare of the group of manipulators $\sum_{i \in M} u_i(S_i)$.
    - (a) and (b) imply that the optimal set S of items for the group maximizes $\sum_{o \in S} \max_{i \in M} u_i(o)$
    - equivalent to solving a manipulation problem for a single manipulator with $o \trianglerighteq o'$ iff $\max_{i \in M} u_i(o) \geq \max_{i \in M} u_i(o')$.
    - optimal manipulation can be found in polynomial time for one non-manipulator; NP-hard for more than one non-manipulator.
- *with item trading and without monetary transfers*: NP-hard, even for two manipulators and one non-manipulator.
- *without item trading nor monetary transfer*: NP-hard, even for two manipulators and zero non-manipulator (!).

More results

- Subgame perfect Nash equilibria for two agents and the alternating sequence are characterized by Kohler and Chandrasekaran (1971).
- (Individual or coalitional) manipulation with *non-additive preferences*: NP-hard even for one nonmanipulator, even for very restricted forms of non-additivity (Bouveret and Lang, 2014)
- One manipulator, several non-manipulators: finding the optimal manipulation falls down to P if the manipulator has binary utilities ($u_i(o) \in \{0, 1\}$)
- Worst-case price of manipulation (loss of social welfare caused by one agent manipulating).
  - for instance, 2 agents + alternating sequence + Borda-induced utilities: the worst-case loss of social welfare converges to $\frac{1}{3}$ when $p \to \infty$ (Bouveret and Lang, 2014)

More topics

- Find an efficient algorithms for computing the optimal picking sequence under the full independence assumption (exact result so far known for 2 agents up to 14 items, and for 3 agents up to 10 items).
- Learning the scoring function for a specific class of problems and a specific population of users
- Which sequences would human decision makers tend to use?