Fair Division of Indivisible Goods on a Graph II

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Advances in Fair Division,
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Fair division of indivisible items

A traditional fair division problem...

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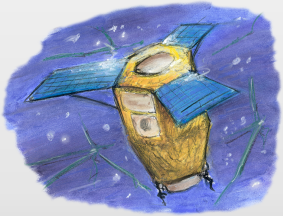
Find

- an allocation $\pi : A \to 2^O$
- such that $\pi(i) \cap \pi(j) = \emptyset$ for every $i \neq j$
- satisfying some fairness and efficiency criteria

A typical example

A common facility to be time-shared...

- a common summer house
- a scientific experimental device
- an Earth observing satellite
- ...



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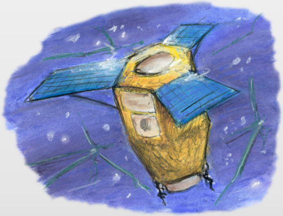

Time-sharing with **predefined timeslots**

A typical example

Predefined timeslots → **indivisible items**

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Fair? Maybe...

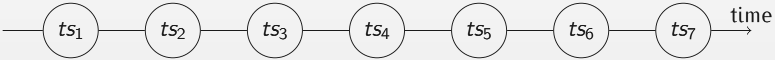
A typical example

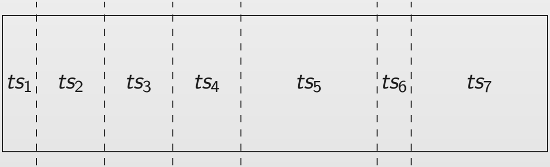
Predefined timeslots → **indivisible items**

Fair? Maybe...
Admissible? Probably not...

Time slots vs cake shares

NB: Can also represent a cake with predefined cut points...





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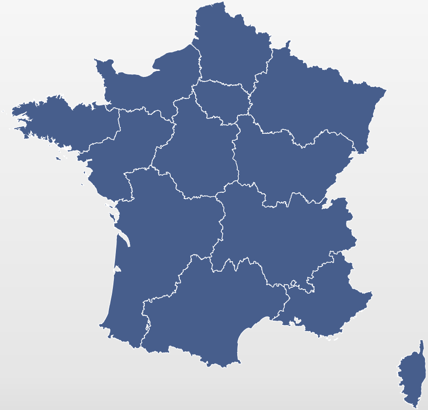

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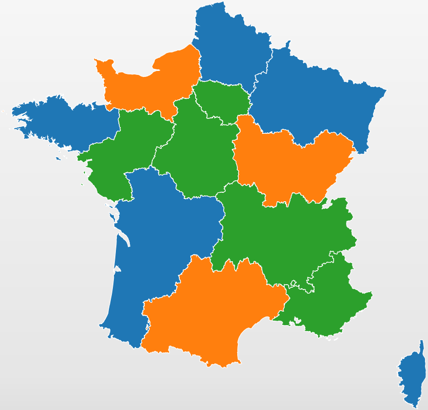

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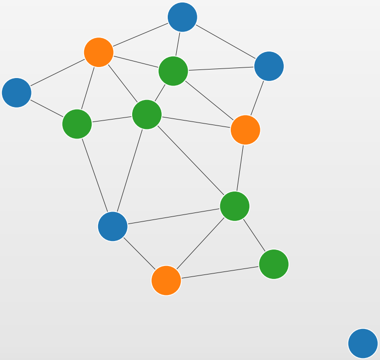

Fair division of a graph

Given

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Fair division of a graph

Given

- a set of indivisible objects $O = \{o_1, \ldots, o_m\}$
- a set of agents $A = \{1, \ldots, n\}$
- each agent has additive preferences on the objects
- a neighbourhood relation $R \subseteq O \times O$ defining a graph of objects G

Find

- an allocation $\pi : A \to 2^O$
- such that $\pi(i) \cap \pi(j) = \emptyset$ for every $i \neq j$
- satisfying some fairness and efficiency criteria
- such that $\pi(i)$ is connected in G for every i

Fairness

The fairness concepts we study:

- **Proportionality:**[1] $u_i(\pi(i)) \geq \frac{1}{n}$ for every i
- **Envy-freeness:**[2] $u_i(\pi(i)) \geq u_i(\pi(j))$ for every (i, j)
- **Max-min share:** $u_i(\pi(i)) \geq u^{MMS}(i)$ for every i, where $u_i^{MMS} = \max_{\vec{\pi}} \min_{j \in N} u_i(\pi_j)$

[1] Equal-division-lower-bound
[2] No-envy

Proportionality

Proportionality: the bad news...

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Proposition

Prop-CFD is NP-complete even if G is a path.

Idea: Reduction from Exact-3-Cover.

Proportionality

Proportionality: the bad news...

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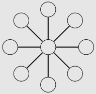
PROP-CFD is NP-complete even if G is a path.

Idea: Reduction from EXACT-3-COVER.

Some good news:

Proposition

PROP-CFD can be solved in polynomial time if G is a star.



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Prop-CFD is XP what is it? with respect to the number of agent types if G is a path.

Idea: dynamic programming algorithm (parameters: number of remaining vertices and number of agents of each type to satisfy)

Proportionality: good news

Proportionality: the good news...

Proposition

PROP-CFD is XP (what is it?) with respect to the number of agent types if G is a path.

Idea: dynamic programming algorithm (parameters: number of remaining vertices and number of agents of each type to satisfy)

Proposition

PROP-CFD is FPT (what is it?) with respect to the number of agents if G is a tree.

Idea: run through all the possible ways of partioning a tree.

Envy-freeness: bad news

Proposition

EF-CFD is NP-complete even if:

- G is a path
- G is a star

Idea:

- Path: (Similar) reduction from Exact-3-Cover
- Star: Reduction from Independent set.

Envy-freeness: good news

Proposition

EF-CFD is XP with respect to the number of agent types if G is a path.

Idea: "Guess" the utility received by each type, and use the previous dynamic programming algorithm (used for proportionality).

Max-min share

Formal definition: $u_i(\pi(i)) \geq u^{MMS}(i)$ for every i, where
$u_i^{MMS} = \max_{\vec{\pi}} \min_{j \in N} u_i(\pi_j)$

More about MMS?

Max-min share

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More about MMS?

Known facts for classical fair division:

- An MMS allocation **almost** always exists
- Counter-examples are rare and intricate [Procaccia and Wang, 2014, Kurokawa et al., 2016]

Kurokawa, D., Procaccia, A. D., and Wang, J. (2016).
When can the maximin share guarantee be guaranteed?
In *AAAI'16*, pages 523–529.

Procaccia, A. D. and Wang, J. (2014).
Fair enough: Guaranteeing approximate maximin shares.
In *ACM EC'14*, pages 675–692.

Max-min share and graphs

Interestingly, as soon as there are connectivity constraints, it is easy to find an instance with no MMS allocation.

Show me the instance

Max-min share and graphs

Proposition

If G is a tree, every agent can compute her MMS share u_i^{MMS} in polynomial time.

Idea: “guess” the value by binary search and “move a knife” along the tree

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Proposition

If G is a tree, an MMS allocation always exists and can be found in polynomial time.

Idea:

- Every agent computes u_i^{MMS}
- We apply a discrete analogue of the last diminisher procedure

Finding an MMS allocation

Intuition of the procedure on a path...

Finding an MMS allocation

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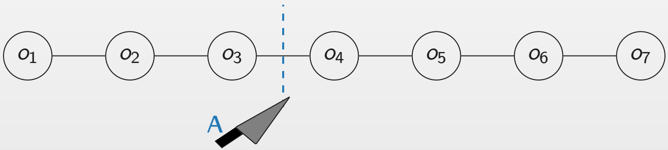

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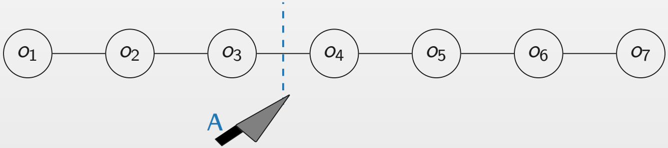

B does nothing

Finding an MMS allocation

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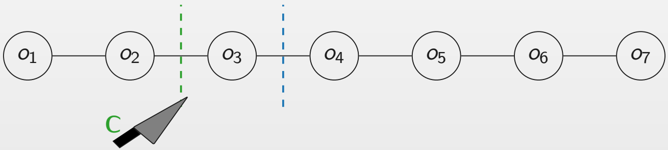

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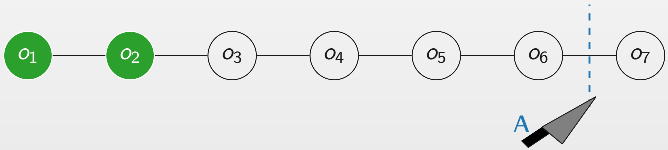

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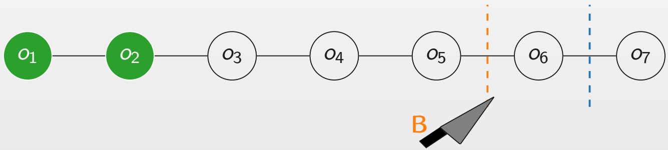

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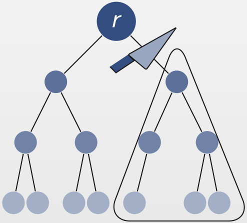
Finding an MMS allocation

Last diminisher on a tree (intuition)...



The first player proposes a bundle.

Finding an MMS allocation

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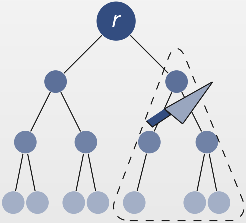
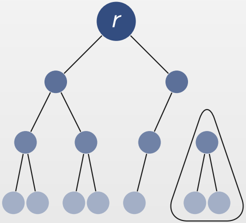


Other players may diminish the bundle.

Finding an MMS allocation

Last diminisher on a tree (intuition)...



The last-diminisher receives the bundle.

Take-away message

- Fair division of indivisible items with connectivity constraints
- Negative (NP-completeness) general results
- But, also positive ones for simple yet interesting cases

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- Path:
  - Proportionality: NP-complete, but XP with respect to the number of agent types and FPT with respect to the number of agents
  - Envy-freeness: NP-complete, but XP with respect to the number of agent types
  - Max-min share: polynomial (and guaranteed to exist)
- Tree:
  - Proportionality: NP-complete, but FPT with respect to the number of agents
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Future work

- Other fairness concepts?
- Other preference representations?
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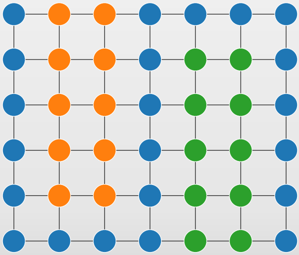

Thank you for your attention

Questions?

Max-min share

Proportionality is nice, but sometimes too demanding for indivisible goods

$\rightarrow$ *e.g.* 2 agents, 1 object

Max-min share

Proportionality is nice, but sometimes too demanding for indivisible goods

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Max-min share (MMS):

- Introduced recently [Budish, 2011]; not so much studied so far.
- **Idea:** in the **cake-cutting** case, proportionality = the best share an agent can hopefully get for sure in a *"I cut, you choose (I choose last)"* game.
- Same game for indivisible goods → MMS.

Budish, E. (2011).
The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes.
Journal of Political Economy, 119(6).

Max-min share

Idea: in the cake-cutting case, proportionality = the best share an agent can hopefully get for sure in a *"I cut, you choose (I choose last)"* game.

Max-min share

The **max-min share** of an agent i is equal to:

$$u_i^{MMS} = \max_{\overrightarrow{\pi}} \min_{j \in N} u_i(\pi_j)$$

An allocation $\overrightarrow{\pi}$ satisfies **max-min share** (MMS) if every agent gets at least her max-min share.

Max-min share: examples

Example: 3 objects $\{1, 2, 3\}$, 2 agents $\{1, 2\}$.

Preferences:

	1	2	3
agent 1	5	4	2
agent 2	4	1	6

Back

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---

Example: 2 agents, 1 object.

$$u_1^{MMS} = u_2^{MMS} = 0 \rightarrow \text{every allocation satisfies MMS!}$$

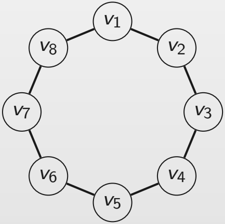
Not very satisfactory, but can we do much better?

Back

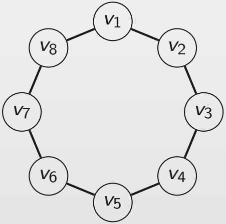
MMS counterexample

	v_1	v_2	v_3	v_4	v_5	v_6	v_7	v_8
Players 1 & 2	1	4	4	1	3	2	2	3
Players 3 & 4	4	4	1	3	2	2	3	1

P_1:



P_2:



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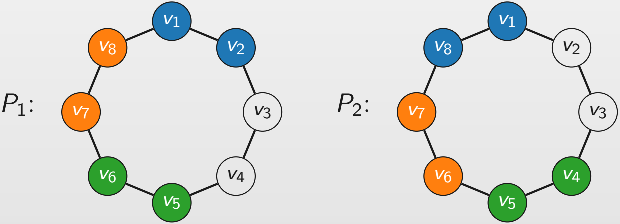

Back

Slice-wise polynomiality

Definition

A problem is **slice-wise polynomial (XP)** with respect to a parameter k if $\exists f$, computable function, s.t. each instance I of this problem can be solved in time $|I|^{f(k)}$.

Intuition: once k is fixed, $f(k)$ can be large, but is fixed. Hence, I can be solved in polynomial time (but the degree of the polynome can be large).

Back

Fixed-parameter tractability

Definition

A problem is **fixed-parameter tractable (FPT)** with respect to a parameter k if $\exists f$, computable function, s.t. each instance I of this problem can be solved in time $f(k) \times poly(|I|)$.

Intuition: once k is fixed, $f(k)$ can be large, but is fixed. I can be solved in polynomial time and the degree of the polynome remains the same for every k.

NB: FPT is strictly contained in XP.

Back