School Choice: basic theory

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March 12, 2019

Last lecture

- House allocation: public ownership
  - serial dictatorship,
  - strategy-proof, neutral, non-bossy and efficient
- Housing markets: private ownership
  - Top trading cycle algorithm
  - the unique core element,
  - the unique competitive allocation
  - efficient, strategy-proof, individually rational

School choice: real-life application of matching

More and more cities around the world use school choice programs:

- school authorities take into account preferences of children and their parents.
- typical goals of school authorities are:
  - (1) efficient placement,
  - (2) fairness of outcomes,
  - (3) easy for participants to understand and use, etc.

Question: could we achieve all these goals? trade-offs?

School choice

When Sönmez was presenting a paper at Carnegie Mellon University, a scholar at the seminar suggested they look at the problems associated with school-choice systems in urban school districts.

Characteristics

- indivisibilities,
- one-sided preferences (students),
- no monetary compensation,
- public ownership assorted with priorities

Outline

- Model
  - Formal model
  - Properties and mechanisms
- Efficiency
  - Priority design
  - School choice with consent

Background

A school district asks parents or students for their preferences

- each school has limited seats,
- all students cannot get their first choice schools for over-demanded schools.
- the district has to reject some students
- efficient, fair and lawsuit-free mechanisms are not trivial,
- design is required

Model

A school choice problem is a triple $(I, S, P, \succ, q)$ where:

- I is a set of students
- S is a set of schools
- P is a list of preferences over $S \cup \{\emptyset\}$
- $\succ$ is list of priorities over I
- q is a vector of positive numbers

Assumption

- *We assume that preferences and priorities are strict,*
- $|I| \leq \sum_{s \in S} q_s$.

Priorities

Where do priorities come from?

- prioritize students living in the walk zone to avoid transportation cost,
- prioritize students who have siblings already attending the school
- exam scores (for student placement to colleges)

Model

- A matching is a function $\mu : I \rightarrow S$ such that for each school s, $|\mu^{-1}(s)| \leq q_s$.
- A mechanism assigns each pair $(P, \succ)$ a matching.

Design goals

- Individual rationality

Definition

A matching μ is individually rational if for each student i

$$\mu(i)\ R_i\ \emptyset$$

- Elimination of justified envy

Definition

A matching μ eliminates justified envy if for each $i \in I$, there is no $j \in I$

$$s\ P_i\ \mu(i), \quad \mu(j) = s \quad \text{and} \quad i \succ_s j.$$

Such a matching is said to be fair.

Design goals

- Non-wastefulness

Definition

A matching μ is non-wasteful if for each student i and each school s

$$s \; P_i \; \mu(i) \quad \Rightarrow \quad |\mu^{-1}(s)| = q_s.$$

- Stability

Definition

A matching is stable if it is individually rational, eliminates justified envy and is non-wasteful.

Design goals

- Strategy-proofness

Definition

A mechanism φ is strategy-proof if for each P and each student i, there is no P'_i such that

$$\varphi_i(P'_i, P_{-i}, \succ)\ P_i\ \varphi_i(P, \succ).$$

Design goals

A priority profile $\succ'$ is an improvement of student i over $\succ$ if

- for each school s
  - the ranking of students in $I \setminus \{i\}$ is the same under $\succ'_s$ and $\succ_s$,
  - student i's ranking did not drop under $\succ'_s$ compared to $\succ_s$,
- for some school s, student i's ranking moved up under $\succ'_s$ compared to $\succ_s$.

Definition

A mechanism φ respects improvements if for each student i, each $\succ$ and each improvement $\succ'$ of student i over $\succ$,

$$\varphi_i(P, \succ') \quad R_i \quad \varphi_i(P, \succ).$$

Student-proposing deferred acceptance

Step 1:

- each student applies to his most preferred acceptable school.
- each school follows its priority and tentatively accepts one at a time its best applicants up to its capacity and rejects the rest.

Step $k, k > 1$

- each student who is rejected at Step $k - 1$ applies to his next acceptable school.
- each school considers the new applicants together with those who are tentatively accepted in the previous step, and follows its priority and accepts one at a time, its best applicants up to its capacity and rejects the rest.

The algorithm terminates when every student is tentatively accepted or has applied to all his acceptable schools.

Deferred acceptance mechanism

Theorem

- *The student-proposing DA is the unique strategy-proof stable matching mechanism,*
- *The DA is the unique stable matching mechanism which respects improvements*
- *The DA Pareto dominates any other stable matching mechanism*
- *The DA is weakly Pareto optimal.*

Proof.

□

DA is not efficient

Example

P_1	P_2	P_3
s_1	s_1	s_2
s_2	s_2	s_1
s_3	s_3	s_3

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
3	1	2
1	2	3
2	3	1

Theorem (Kesten, 2010)

There is no strategy-proof and efficient mechanism which selects an efficient and stable matching whenever such a matching exists.

DA is not efficient

Example

P_1	P_2	P_3
s_1	s_1	s_2
s_2	s_2	s_1
s_3	s_3	s_3

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
3	1	2
1	2	3
2	3	1

Theorem (Kesten, 2010)

There is no strategy-proof and efficient mechanism which selects an efficient and stable matching whenever such a matching exists.

Top trading cycle (TTC) mechanism

Step 1

- Each student points to his most preferred acceptable school. Each school points to the student with the highest priority,
- each student in each cycle is assigned to the school he is pointing to and removed, while the capacity of each of these schools is reduced by one.

Step k, $k > 1$*:*

- Each remaining student points to his next most preferred acceptable school. Each school with remaining seats points to the student with the highest priority (there is a cycle!)
- each student in each cycle is assigned to the school he is pointing to and removed, while the capacity of each of these schools is reduced by one.

TTC

The algorithm terminates when no school or student remains or no student finds any remaining school acceptable.

Theorem

The TTC mechanism is strategy-proof, efficient and individually rational. The TTC mechanism is not stable.

Example

P_1	P_2	P_3
s_1	s_1	s_2
s_2	s_2	s_1
s_3	s_3	s_3

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
3	1	2
1	2	3
2	3	1

TTC not adopted in practice

TTC is used for the first time in New Orleans in 2012.

A memorandum of the *Boston Public School* board.

"[TTCs] *trading shifts the emphasis onto the priority and away from the goals BPS is trying to achieve by granting these priorities in the first place.*"

TTC hasn't received much popularity because school seats are owned by districts while the mechanism grants seats to students.

Boston mechanism: a mechanism from practice

Step 1:

- Each student applies to his first acceptable choice school,
- each school follows its priority and immediately accepts one at a time its best applicants until up to its capacity and rejects the remaining applicants

Step $k, k > 1$*:*

- each student who is rejected at Step $k - 1$ applies to his k'th acceptable choice,
- each college follows its priority and immediately accepts its best new applicants up to its remaining seats.

The algorithm terminates when each student has been accepted or has been rejected by all his acceptable schools or no school has remaining seats.

The Boston mechanism

Theorem

- *The Boston mechanism is not strategy-proof,*
- *the Boston mechanism is Pareto efficient*
- *the Boston mechanism is not stable*
- *the Boston mechanism respects improvements*
- *the Boston mechanism maximizes the number of students who receive their first choice, second choice, etc.*

DA is a big winner

Theorem (Kesten, 2010)

There is no strategy-proof mechanism which Pareto dominates DA.

Proof.

□

DA is a big winner

Properties	DA	TTC	Boston
Stable	yes	no	no
Strategy-proof	yes	yes	no
efficient	no	yes	yes
respects improvement	yes	yes	yes
weakly efficient	yes	yes	yes
optimal stable	yes	no	no

DA is a big winner

Consider the game of the Boston mechanism:

Theorem (Sonmez & Ergin, 2008)

The set of Nash equilibrium outcomes of the Boston mechanism is equivalent to the set of stable matchings.

Corollary

The dominant-strategy outcome of DA Pareto dominates any other Nash equilibrium outcome of the Boston mechanism.

Proof.

□

The inefficiency in DA can be large

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}	P_{i_7}	P_{i_8}	P_{i_9}	$P_{i_{10}}$	$P_{i_{11}}$	$P_{i_{12}}$
$\boxed{s_1}$	$\boxed{s_1}$	s_1	$\boxed{s_2}$	$\boxed{s_2}$	s_2	$\boxed{s_3}$	$\boxed{s_3}$	s_3	$\boxed{s_4}$	$\boxed{s_4}$	s_4
s_2	s_3	s_3	s_3	s_4	s_1	s_4	s_2	s_4	s_1	s_1	s_2
s_4	s_4	s_2	s_1	s_3	s_4	s_2	s_1	s_1	s_3	s_2	s_3
$\underline{s_3}$	$\underline{s_2}$	s_4	$\underline{s_4}$	$\underline{s_1}$	s_3	$\underline{s_1}$	$\underline{s_4}$	s_2	$\underline{s_2}$	$\underline{s_3}$	s_1
s_5	s_5	$\underline{\boxed{s_5}}$	s_5	s_5	$\underline{\boxed{s_5}}$	s_5	s_5	$\underline{\boxed{s_5}}$	s_5	s_5	$\underline{\boxed{s_5}}$

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
i_5	i_2	i_{11}	i_4	i_{12}
i_7	i_{10}	i_1	i_8	i_9
i_{12}	i_9	i_6	i_3	i_6
i_4	i_7	i_{10}	i_1	i_3
i_8	i_{11}	i_5	i_2	$\vdots$
i_9	i_3	i_{12}	i_6	
i_{10}	i_1	i_4	i_7	
i_{11}	i_8	i_2	i_5	
i_6	i_{12}	i_3	i_9	
i_1	i_4	i_7	i_{10}	
i_2	i_5	i_8	i_{11}	

How do we deal with inefficiency in DA?

Two approaches:

- Priority design (Ergin, 2002),
- Consenting (Kesten, 2010)

Priority design

Question: are there priority structures for which DA is efficient for each preference profile?

Definition

A cycle for a priority structure $\succ$ is a triple (i, j, k) of students and a couple (s_1, s_2) of schools such that

- Cycle condition (CC): $i \succ_{s_1} j \succ_{s_1} k \succ_{s_2} i$.
- Scarcity condition (SC): there is two disjoint sets $I_1, I_2 \subset I \setminus \{i, j, k\}$ such that $|I_1| = q_{s_1} - 1$, $|I_2| = q_{s_2} - 1$, for each $\ell \in I_1$, $\ell \succ_{s_1} j$ and for each $\ell \in I_2$, $\ell \succ_{s_2} k$.

A priority structure $\succ$ is acyclic if it has no cycle.

Priority design

Theorem (Ergin, 2002)

Given a priority structure $\succ$, the following are equivalent:

- *for each P, $DA(P, \succ)$ is efficient*
- *$DA(., \succ)$ is group strategy-proof*
- *$\succ$ is acyclic.*

Acyclic priority structures are stringent

Theorem (Ergin, 2002)

A priority structure is acyclic if, and only, if the priority rankings of any pair of schools is such that the position of any student ranked below the sum of their capacities differs by more than one.

Efficiency cost of DA: consenting

- Respecting priorities has a cost in terms of efficiency.
- Solution: ask students for their permissions to violate their priorities whenever this could help others.
- No other student's priority could be violated.

School choice with consent

Efficiency Adjusted Deferred Acceptance.

Example

P_1	P_2	P_3
s_1	s_1	s_2
s_2	s_2	s_1
s_3	s_3	s_3

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
3	1	2
1	2	3
2	3	1

Definition

Student i is an interrupter of Step t of DA if

- student i has been tentatively accepted by school s at Step $t' < t$ of DA,
- has been rejected from school s at Step t and
- some student has rejected from school s at a Step $r \in \{t', \ldots, t-1\}$

School choice with consent

Efficiency Adjusted Deferred Acceptance.

Example

P_1	P_2	P_3
s_1	s_1	s_2
s_2	s_2	s_1
s_3	s_3	s_3

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
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Neutralizing interrupters: a challenge!

Example (Student i_1 and i_2 are interrupters)

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
i_3	i_1	$\vdots$
i_1	i_2	
i_2	i_3	

P_{i_1}	P_{i_2}	P_{i_3}
$\boxed{s_1}$	s_1	$\boxed{s_2}$
$\underline{s_2}$	s_2	$\underline{s_1}$
s_3	$\underline{\boxed{s_3}}$	s_3

Step	s_1	s_2	s_3
1	$\boxed{i_1}$, i_2	$\boxed{i_3}$	
2	$\boxed{i_1}$	i_3, $\boxed{i_2}$	
3	i_1, $\boxed{i_3}$	$\boxed{i_2}$	
4	$\boxed{i_3}$	i_2, $\boxed{i_1}$	
5	$\boxed{i_3}$	$\boxed{i_1}$	$\boxed{i_2}$

Efficiency Adjusted Deferred Acceptance

Example

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
i_2	i_3	i_1	i_4	$\vdots$
i_1	i_6	i_6	i_3	
i_5	i_4	i_2	i_6	
i_6	i_1	i_3	$\vdots$	
i_4	$\vdots$	$\vdots$		
i_3				

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}
$\boxed{s_2}$	$\boxed{s_3}$	s_3	$\boxed{s_1}$	s_1	s_4
s_1	$\underline{s_1}$	$\boxed{s_4}$	s_2	$\underline{\boxed{s_5}}$	s_1
$\underline{s_3}$	s_5	$\underline{s_2}$	$\underline{s_4}$	$\vdots$	s_3
$\vdots$	$\vdots$	$\vdots$			s_2
					$\underline{\boxed{s_5}}$

Example

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_5}$, i_4	$\boxed{i_1}$	$\boxed{i_2}$, i_3	$\boxed{i_6}$	
2	$\vdots$	i_1, $\boxed{i_4}$	$\vdots$	i_6, $\boxed{i_3}$	
3	i_5, i_6, $\boxed{i_1}$	$\vdots$		$\vdots$	
4	$\vdots$		i_2, $\boxed{i_6}$		$\boxed{i_5}$
5	$\boxed{i_2}$, i_1		$\vdots$		$\vdots$
6	$\vdots$		$\boxed{i_1}$, i_6		
7		$\boxed{i_6}$, i_4	$\vdots$		
8		$\vdots$		$\boxed{i_4}$, i_3	
9		i_6, $\boxed{i_3}$		$\vdots$	
10	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_1}$	$\boxed{i_4}$	$\boxed{i_5, i_6}$

Student i_6 is the last interrupter for school s_3

Example

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_5}, i_4$	$\boxed{i_1}$	$\boxed{i_2}, i_3$	$\boxed{i_6}$	
2	$\vdots$	$i_1, \boxed{i_4}$	$\vdots$	$i_6, \boxed{i_3}$	
3	$i_5, i_6, \boxed{i_1}$	$\vdots$		$\vdots$	
4	$\vdots$		$i_2, \boxed{i_6}$		$\boxed{i_5}$
5	$\boxed{i_2}, i_1$		$\vdots$		$\vdots$
6	$\vdots$		$\boxed{i_1}, i_6$		
7	$\boxed{i_2}$	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_3}$	$\boxed{i_5, i_6}$

Student i_5 is the last interrupter for school s_1

Example

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_5}, i_4$	$\boxed{i_1}$	$\boxed{i_2}, i_3$	$\boxed{i_6}$	
2	$\vdots$	$i_1, \boxed{i_4}$	$\vdots$	$i_6, \boxed{i_3}$	
3	$i_5, i_6, \boxed{i_1}$	$\vdots$		$\vdots$	
4	$\boxed{i_1}$	$\boxed{i_4}$	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_5, i_6}$

Student i_6 is the last interrupter for school s_1

Example

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_5}, i_4$	$\boxed{i_1}$	$\boxed{i_2}, i_3$	$\boxed{i_6}$	
2	$\vdots$	$i_1, \boxed{i_4}$	$\vdots$	$i_6, \boxed{i_3}$	
3	$i_5, i_6, \boxed{i_1}$	$\vdots$		$\vdots$	
4	$\boxed{i_1}$	$\boxed{i_4}$	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_5, i_6}$

No interrupter

Example

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_2}, i_3$	$\boxed{i_6}$	$\boxed{i_5}$
2	$\vdots$	$\vdots$	$\vdots$	$i_6, \boxed{i_3}$	$\vdots$
3	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_5, i_6}$

Efficiency Adjusted Deferred Acceptance

Round 0: Run the DA for $(P, \succ)$.

Round $k, k > 1$.

- Identify the last step of Round $k - 1$ of DA in which a consenting interrupter is rejected.
- identify all consenting interrupters of that step.
- for each of such students remove the respective school from the interrupter's preferences
- run DA with the new preferences

The algorithm stops when there are no more consenting interrupters.

School choice with consent

Theorem (Kesten, 2010)

When some students consent, the EADA weakly Pareto dominates DA. When all students consent, the EADA is Pareto efficient.

Simplified EADA:

Example (Continued)

School choice with consent

Theorem (Kesten, 2010)

When some students consent, the EADA weakly Pareto dominates DA. When all students consent, the EADA is Pareto efficient.

Simplified EADA:

Example (Continued)

School choice with consent: simplified EADA

Round 0: Run DA for the problem $(P, \succ)$

Round $k, > 1$*:* There are three steps

- identify the schools which are underdemanded at Round $k-1$ of DA and remove these schools and students who are matched to them.
- for each removed student who does not consent, each remaining school s that student i desires and each remaining student j such that $i \succ_s j$, remove school s from j's preference.
- Rerun DA for the subproblem that consists of only the remaining schools and students.

Stop when all schools are removed.

School choice with consent: simplified EADA

Theorem (Tang and Yu, 2014)

Under the simplified EADA, the assignment of any student does not change whether she consent or not. No student has the incentive not to consent.

Theorem (Tang & Yu, 2014)

The simplified EADA is equivalent to EADA and is thus Pareto efficient when all students consent.

Proof.

□

Next lecture: school choice!

Development of new theory: sincere and sophisticated students in the Boston mechanism.