House allocation & Housing market

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Last lecture

- the Deferred Acceptance algorithm (DA) produces a stable matching:
  - the most preferred stable matching for the proposing side,
  - the least preferred stable matching for the receiving side,
- DA is strategyproof for the proposing side, but not for the receiving side,
- we cannot have, in general, strategyproofness for both sides and stability,
- any man or woman who did not receive his side optimal-stable's mate at a stable matching mechanism can misrepresent his preferences in such a way to receive it.

Blocking lemma

Lemma (Blocking lemma)

Let μ be an individually rational matching with respect to strict preferences P, and let M' be the set of all men who prefer μ to μ_M.

If M' is nonempty, there is a pair (m, w) that blocks μ such that

$$m \notin M' \text{ and } w \in \mu(M').$$

Proof.

□

House allocation & Housing markets: real-life applications

- Organ allocation,
- domitory room allocation at universities,
- school choice programs,
- parking space,
- office allocation,
- allocation of irrigated parcels,
- course allocation.

Characteristics

- indivisibilities,
- one-sided preferences,
- no monetary compensation
- public or private ownership

Outline

- House allocation
  - Serial dictatorship
  - Characterization
- Housing market
  - Top trading cycle
  - Competitive equilibrium
  - Characterization

Model

A house allocation problem (Hylland & Zeckhauser, JPE 1979) is a triple $(N, H, \succ)$ where:

- N is a set of agents
- H is a set of houses
- $\succ$ is a list of preferences over H

Assumption

- *We assume that preferences are strict,*
- $|H| = |I|$.

Model

An allocation is a bijection $a : I \to H$ where $a(i)$ is the house assigned to agent i.

Efficient house allocation

- After matching, and if allowed, agents can swap houses
- improving agents' welfare is desirable.

Definition

An allocation a is Pareto efficient if there is no other allocation b such that

$$\text{for each agent } i, \; b(i) \succeq_i a(i),$$

and

$$\text{for some agent } i, \; b(i) \succ_i a(i)$$

Which mechanisms are recommendable?

Definition

A mechanism φ maps the set $\mathcal{P}^I$ of preference profiles to the set $\mathcal{M}$ of matchings.

- A mechanism φ is efficient if for preference profile $\succ$, $\varphi(\succ)$ is efficient.
- A mechanism φ is strategy-proof if for each $\succ$, each agent i there is no $\succ'_i$ such that

$$\varphi_i(\succ'_i, \succ_{-i}) \succ_i \varphi_i(\succ).$$

Which mechanisms are recommendable?

Definition

- A mechanism φ is neutral if for each permutation (relabelling) $\pi : H \to H$, each preference profile $\succ$, for each agent i,

$$\varphi_i(\pi(\succ)) = \pi(\varphi_i(\succ)).$$

- A mechanism φ is non-bossy if for each preference profile $\succ$, agent i and each $\succ_i'$

$$\varphi_i(\succ_i', \succ_{-i}) = \varphi_i(\succ) \Rightarrow \varphi(\succ_i', \succ_{-i}) = \varphi(\succ).$$

SD as a unique strategy-proof, neutral and non-bossy mechanism

Theorem (Svesson, 1999)

A mechanism is strategy-proof, neutral and non-bossy if, and only if, it is serial dictatorship.

Definition (Serial dictatorship)

There is an ordering of the agents. For each preference profile $\succ$, the agent ordered first picks his most preferred house, the agent ordered second picks his most preferred house among the remaining houses etc.

Note that SD is efficient.

SD as a unique strategy-proof, neutral and non-bossy mechanism

Proof.

- Neutrality: calibration for maximum conflicting preferences
  For each preference profile $\succ$ in which agents have the same preference, the same agent receives his most preferred house, the same agent receives his second most preferred house, etc.
- Strategy-proofness and non-bossiness:
  Let $\succ$ be a preference profile. We claim that $\varphi(\succ) = SD^f(\succ)$.
  Let a_1 be the house allocated to agent $f(1)$ under $SD^f(\succ)$, a_2 the house allocated to agent $f(2)$ under $SD^f(\succ)$ etc.
  Let $\succ^*$ be the preference profile where each agent orders house a_1 first, house a_2 second, etc.

□

Model

A housing market is a couple $(N, \succ)$

- N is a set of agents
- Each agent i has one house denoted i
- $\succ$ is a list of preferences such that, for each i, $\succ_i$ is agent i's preference over N.

Assumption

We assume that agents have strict preferences.

Core of a market

Definition

An allocation b weakly dominates another a via a coalition $S \subset N$ if

- $b(S) = S$,
- for each $i \in S$, $b(i) \succeq_i a(i)$
- for some $i \in S$, $b(i) \succ_i a(i)$.

The core of a market is the set of allocations which are not dominated.

The core is nonempty

Theorem

The core of the market $\succ$ contains one allocation: it is the allocation obtained by the top trading cycles algorithm.

Top Trading Cycles (TTC) algorithm (attributed to Gale)

- **Step 1:**
  - Each agent points to his most preferred house, (there is cycle!)
  - each agent in each cycle is assigned to the house he is pointing to and these agents and the corresponding houses are removed.

The core is nonempty

- **Step 2:**
  - Each agent points to his most preferred house among those that remain, (there is cycle!)
  - each agent in each cycle is assigned to the house he is pointing to and these agents and the corresponding houses are removed.

The core is nonempty

- **Step k:**
  - Each agent points to his most preferred house among those that remain, (there is cycle!)
  - each agent in each cycle is assigned to the house he is pointing to and these agents and the corresponding houses are removed.

The algorithm terminates when no agent and a house remain.

HW: Prove that the algorithm is well-defined.

TTC: example

$\succ_1$	$\succ_2$	$\succ_3$	$\succ_4$	$\succ_5$
2	3	1	1	1
4	2	4	2	2
3	1	5	4	4
1	5	3	3	5
5	4	2	5	3

Proof of the theorem.

□

Competitive equilibria

Definition

A pair (a, p) of an allocation a and a n-vector of positive and non-zero prices is a competitive equilibrium for the market $\succ$ if

- for each agent i, $p_{a(i)} \leq p_i$ (budget constraint)
- for each $i, j \in N$, $j \succ_i a(i) \Rightarrow p_j > p_i$ (maximizing utility).

Theorem (Roth & Postlewaite, 1977)

For each market, there is a unique competitive equilibrium allocation: it is the allocation given by TTC.

Proof.

- TTC allocation is an equilibrium allocation
- every competitive equilibrium allocation coincides with TTC (HW)

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TTC is efficient, IR and strategy-proof

Definition

A mechanism φ is individually rational if for each market $\succ$ and each agent i,

$$\varphi_i(\succ) \succeq_i i.$$

Theorem

TTC is efficient and individually rational.

Theorem (Roth, 1982)

TTC mechanism is strategy-proof.

Proof.

□

TTC is the unique efficient, IR and strategy-proof mechanism

Theorem (Ma, 1994)

A mechanism is strategy-proof, efficient and IR if, and only, if it is TTC.

Proof.

□

Take-away

- Serial dictatorship for house allocation (when the houses are not owned by any agent)
- TTC for housing market (when houses are owned by individual agents).

Next lecture: school choice!
Application of our theory and development of new theory.