# Elements of the theory of perfect information games: Determinacy, equilibrium, and subgame perfection

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**Course summary**. Perfect information games are strategic situations where, at any stage of the game, each player knows what actions have been taken prior to that stage, i.e. she has perfect information about past play. Games of this type appear in a variety of disciplines: in descriptive set theory, logic, topology, analysis, and other areas of mathematics, computer science, and economics, just to name a few. For this reason, the theory of perfect information games is vast and multifaceted. The course touches upon but a few aspects of the theory.

I. DETERMINACY. In the first part of the course we consider perfect information games played by two players, called I and II. The two players consecutively choose elements (called actions or moves) from a set $A$, player I moving at even stages, player II at odd stages. At each stage of the game, the players know exactly what choices have been made in the past:

$$\begin{array}{lccccc} \text{I} & x_0 & & x_2 & & \cdots \\ \text{II} & & x_1 & & x_3 & \cdots \end{array}$$

In this way, the two players jointly produce a run of the game, $x = (x_0, x_1, \dots)$, an infinite sequence of actions; a run is thus a point of $A^{\mathbb{N}}$. The objective of player I is represented by a subset $W \subseteq A^{\mathbb{N}}$ of runs, called player I's winning set. The run $x$ is won by player I if it is an element of $W$; if it is not, the run $x$ is won by player II.

A winning strategy of player I is a strategy that guarantees the run of the game to be in the winning set $W$, however player II might play. Likewise, player II's winning strategy is a strategy that ensures the run of the game to be outside the winning set $W$, irrespective of the moves of player I. The game is said to be determined if either player I has a winning strategy, or if player II does. The central question in the theory of such games is what kinds of games are determined.

We will study the construction of a non-determined game, and the proof that closed games (and open games) are determined. Part I of the course culminates with the discussion of Martin's celebrated result on determinacy of Borel games.

II. EQUILIBRIUM. In the second part of the course we enlarge the class of perfect information games under consideration.

Firstly, we consider situations where player I's objective is no longer represented by a winning set, but by a payoff function, which one can think of specifying an amount of money player II has to pay player I, depending on the realized run of the game. Player I's objective is to maximize the payoff, player II's is to minimize it. This model is a zero-sum (perfect information) game. The key notions for zero-sum games are that of an upper value and the lower value. The lower value is the highest payoff that player I can guarantee to receive. The upper value is the lowest payoff that player II can force upon player I. When the two are equal, the game is said to have a value.

A zero-sum perfect information game with a bounded and Borel measurable payoff function has a value, say $v$, and both players have $\epsilon$-optimal strategies: player I's $\epsilon$-optimal strategy guarantees player I to receive no less than $v - \epsilon$, while player II's $\epsilon$-optimal strategy guarantees player II to pay no more than $v + \epsilon$.

Secondly, we consider games with an arbitrary number of players, each player striving to maximize her own payoff function. The first solution concept we examine is ($\epsilon$)-equilibrium: a strategy profile from which no player has an incentive to unilaterally deviate. The key result is that a perfect information game with bounded Borel measurable payoff functions admits an $\epsilon$-equilibrium, for each $\epsilon > 0$.

III. SUBGAME PERFECTION. Subgame perfect equilibrium is a refinement of equilibrium. The notion is motivated by the observation that equilibria in perfect information games might involve the so-called non-credible threats: threats of punishment that are in no one's interest to carry out.

We will see that not all perfect information games have a subgame perfect $\epsilon$-equilibrium. We start by analyzing counterexamples to the existence subgame perfect $\epsilon$-equilibrium, and continue to discussing several sufficient conditions for its existence. We will also discuss the one-deviation principle, and a related notion of a one-deviation equilibrium.

**Who is this course for?** For anyone interested in mathematical game theory. While the course is meant to be largely self-contained, it does require certain aptitude for abstract thinking and sure-footedness with mathematical reasoning. Knowledge of basic topology (open/closed sets, Borel sets) would be helpful but not indispensable.

**Selected references**.

PART I

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PART II

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PART III

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